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Analytical and Computer-Assisted Proofs in Incompressible Fluids

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Resumen y Conclusiones

En esta memoria mostramos resultados sobre ecuaciones que provienen de la mecánica de fluidos incompresibles. En particular, tratamos problemas de frontera libre, que modelizan la evolución de la interfase que separa dos fluidos inmiscibles de diferentes densidades. Nuestro interés se ha centrado en el análisis de la formación de singularidades en tiempo finito.

Uno de los problemas estudiados es el movimiento de las olas (*water waves* en inglés), esto es, cuando los fluidos son agua y aire, que en nuestro caso asumiremos que tienen densidades iguales a 1 y a 0 respectivamente. Asumiremos también que dichos fluidos son irrotacionales y que la vorticidad (el rotacional de la velocidad) está concentrada en la interfase. Estas ecuaciones también son conocidas como el problema de frontera libre para las ecuaciones de Euler incompresibles. El otro problema que presentaremos es el problema de Muskat, que modeliza el comportamiento de la interfase entre dos fluidos incompresibles en un medio poroso, donde la ecuación del movimiento viene dada por la ley de Darcy.

La estructura de esta disertación presenta dos partes claramente diferenciadas: la primera (capítulos 1 a 3) corresponde a las técnicas clásicas del análisis y de las ecuaciones en derivadas parciales mientras que la segunda (capítulos 4 a 6) utiliza el ordenador para probar (de forma rigurosa) los teoremas.

La primera parte está dividida en tres capítulos. En el primero se realiza una introducción y un repaso del estado de la cuestión del problema de las *water waves*.

En el segundo se presenta el resultado central de la tesis: la demostración de la formación de singularidades de tipo *splash* y *splat*, que corresponden físicamente al momento en el que la ola rompe al chocar consigo misma en un único punto (*splash*) o en una curva (*splat*). Los ingredientes esenciales de la prueba comprenden una desingularización mediante una transformación conforme del dominio en el que ocurre la singularidad y estimaciones de energía para el problema en el nuevo dominio desingularizado con el fin de obtener existencia local. Dicha existencia local se demuestra tanto en el espacio de funciones analíticas como en espacios de Sobolev. Los resultados de este capítulo han sido publicados en [19] y [20].

En el tercero se estudia la influencia de la tensión superficial en el modelo y si ésta es capaz o no de prevenir la aparición de singularidades *splash* o *splat*. Aquí demostramos que dichas singularidades pueden surgir incluso en el caso en que haya tensión superficial. El estudio de este caso se puede encontrar publicado en [18].

La segunda parte se fragmenta en tres capítulos también. En el primero se realiza una introducción de la aritmética de intervalos y las pruebas asistidas por ordenador, haciendo énfasis en su uso en el marco del análisis y de las ecuaciones en derivadas parciales.

En el segundo se describe un posible esquema de demostración del siguiente resultado: existen condiciones iniciales que inicialmente son un grafo, en un tiempo finito desarrollan una singularidad de tipo *turning* (esto es, que la interfase deja de ser un grafo) y finalmente colapsan en una singularidad de tipo *splash*. La primera parte del resultado fue demostrada por Castro, Córdoba, Fefferman, Gancedo y López-Fernández en [22] mientras que la segunda es el capítulo dos de la primera parte de esta memoria. No es evidente la conexión entre ambos resultados a priori puesto que no se sabe que los conjuntos de soluciones que verifican cada uno de los teoremas compartan elementos. En este capítulo se presentan resultados parciales en esta dirección y se sugiere cómo se podría completar el resto de la demostración mediante el uso extensivo de técnicas en las que predomina el uso del ordenador como herramienta rigurosa de demostración.

En el tercer capítulo se usan las técnicas anteriores para demostrar rigurosamente una serie de teoremas sobre la formación de singularidades turning para el problema de Muskat. Castro, Córdoba, Fefferman, Gancedo y López-Fernández probaron en [22] que existen datos iniciales que giran, pasando al régimen inestable. En nuestro caso realizamos un estudio sobre las condiciones en las que se puede dar el giro comparando diversos modelos: el modelo confinado (en el que los fluidos se encuentran situados entre dos "tapas" situadas a una altura finita) y el no confinado, así como los casos en los que el medio presenta un salto de permeabilidades (modelo no homogéneo). El resultado de dicho trabajo se encuentra en [52].

Por último, se adjuntan los códigos correspondientes a las simulaciones numéricas de la primera parte y la prueba asistida por ordenador de la segunda parte en los apéndices A y B respectivamente.

Abstract and Conclusions

This dissertation is devoted to the study of equations arising in the field of fluid mechanics, more precisely incompressible fluids. In particular, we consider free boundary problems, which model the evolution of an interface between two immiscible fluids with different densities. Attention is focused on the analysis of finite time singularity formation.

One of the studied problems is the so-called *water waves* problem, which approximates the behaviour of the sea waves, (i.e. when the fluids are water and air, which in this case we think of having densities one and zero respectively). We will also assume that these fluids are irrotational and that the vorticity (the curl of the velocity) is concentrated on the interface. These equations also receive the name of the free boundary incompressible Euler equations. The other one is the Muskat problem, which models the behaviour of the interface between two incompressible fluids in a porous medium. In this case, the equation of movement is given by Darcy's Law.

The structure of this work has two highly differentiated parts: the first part (chapters 1 to 3) corresponds to the classical techniques coming from analysis and partial differential equations while the second (chapters 4 to 6) employs the computer to rigorously prove the theorems.

The first part is divided into three chapters. The first one consists of an introduction and a brief survey about the state of the art concerning the water waves problem.

In the second one the main result of this thesis is presented: the proof of the formation of *splash* and *splat* singularities, which physically correspond to the moment in which the wave turns and breaks down while self-intersecting, either in a single point or along an arc. The main ingredients of the proof are a desingularization of the domain in which the singularity occurs by means of a conformal map and energy estimates for the new problem in the desingularized domain in order to obtain a local existence theorem. The space in which we can prove local existence can be either the space of analytic functions or a Sobolev space. The results of this chapter have been published in [19] and [20].

In the third chapter the influence of surface tension in the model is studied. More specifically, an answer to the question whether surface tension can prevent the appearance of splash or splat singularities is given. Such singularities can occur even when surface tension is present. This study can be found in [18].

The second part is fragmented into three chapters. In the first one an introduction to interval arithmetics and computer-assisted proofs is made, emphasizing in their use in the framework of analysis and partial differential equations.

In the second one, we describe a possible approach to a proof of the following result: there exist initial conditions that initially can be written as a graph, develop a *turning* singularity (this means the interface stops being a graph) in finite time and finally collapse into a splash singularity. The first part of the result was proved by Castro, Córdoba, Fefferman, Gancedo and López-Fernández in [22], while the second can be found in the second chapter of the first part of this manuscript. The connection between these two results is not evident since a priori it is not known whether the sets of solutions to both theorems have common elements. In this

chapter, partial results (both rigorous and non rigorous) are presented and suggestions about how the full proof could be completed are made. In our case, this completion is based in techniques in which the use of the computer as a rigorous theorem prover tool predominates.

In the third chapter an example illustrating how the previous techniques can be put into practice is made. We prove several theorems concerning the formation of turning singularities for the Muskat problem. Castro, Córdoba, Fefferman, Gancedo and López-Fernández proved in [22] that there exists a class of initial data that develops turning singularities for the Muskat problem, moving into the unstable regime. In our case, we carry out a study about the conditions under which the turning can be created by comparing different models: the confined model (in which the fluids are hold between two fixed boundaries situated at a finite height) and the non-confined model, as well as cases in which the medium presents a jump on the permeabilities. The result of this work appears in [52].

Finally, the codes corresponding to the numerical simulations of the first part and the rigorous computer-assisted proof of the second part can be found in appendices A and B respectively.

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Sin duda alguna, esta tesis no habría sido posible sin mi director, Diego Córdoba. Me gustaría agradecerle todo el tiempo y la dedicación que ha puesto, pero en especial su capacidad de motivación para sacar lo mejor de mí, aún (y sobre todo) cuando uno piensa que las cosas no van a salir como espera. Le doy las gracias por todas las oportunidades que me ha dado, porque la puerta de su despacho siempre estuvo abierta para mí y por su filosofía de trabajo, que han hecho de estos cuatro últimos años una época memorable.

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Y a mi madre, de la que estoy orgulloso, que siempre ha estado ahí.

A mi familia

Contents

Chapter 1

Introduction to the Water Waves problem

1.1 Statement of the Problem

The water wave equations (or 2D incompressible free boundary Euler equations) describe a system consisting of a connected water region $\Omega(t) \subset \mathbb{R}^2$ and a vacuum region $\mathbb{R}^2 \setminus \Omega(t)$, evolving as a function of time t, and separated by a smooth interface

$$\partial\Omega(t) = \{z(\alpha, t) : \alpha \in \mathbb{R}\}.$$

We write $\Omega^1(t) = \mathbb{R}^2 \setminus \Omega(t)$, $\Omega^2(t) = \Omega(t)$. The fluid velocity $v(x, y, t) \in \mathbb{R}^2$ and the pressure $p(x, y, t) \in \mathbb{R}$ are defined for $(x, y) \in \Omega(t)$. The fluid is assumed to be incompressible and irrotational

$$\nabla \cdot v = 0, \quad \text{curl } v = 0 \quad \text{ in } \Omega(t), \tag{1.1}$$

and to satisfy the 2D Euler equation

$$[\partial_t + (v \cdot \nabla_x)]v(x, y, t) = -\nabla p(x, y, t) - (0, g) \quad \text{ in } \Omega(t), \tag{1.2}$$

where $g > 0$ accounts for the gravitational acceleration.

Neglecting surface tension, we assume that the pressure satisfies

$$p = p^*(t) \quad \text{ at } \partial\Omega(t), \text{ where } p^*(t) \text{ is a function of } t \text{ alone.} \tag{1.3}$$

Finally, we assume that the interface moves with the fluid, i.e.,

$$\partial_t z(\alpha, t) = v(z(\alpha, t), t) + c^{\#}(\alpha, t)\partial_\alpha z(\alpha, t), \tag{1.4}$$

where $c^{\#}(\alpha, t)$ is an arbitrary smooth function of α, t (the choice of $c^{\#}$ affects only the parametrization of $\partial\Omega(t)$) and $z(\alpha, t) = (z_1(\alpha, t), z_2(\alpha, t))$.

At an initial time t_0, we specify the fluid region $\Omega(t_0)$ and the velocity $v(x, y, t_0)$ ($(x, y) \in \Omega(t_0)$), subject to the constraint (1.1). We then solve equations (1.1-1.4) with the given initial conditions, and we ask whether a singularity can form in finite time from an initially smooth velocity $v(\cdot, t_0)$ and fluid interface $\partial\Omega(t_0)$. In this part of the thesis, we prove that water waves in two space dimensions can form a singularity in finite time by either of two simple, natural scenarios, which we call a "splash" and a "splat".

The water wave problem comes in three flavors:

- Asymptotically Flat: We may demand that $z(\alpha, t) - (\alpha, 0) \to 0$ as $\alpha \to \pm\infty$.
- Periodic: We may instead demand that $z(\alpha, t) - (\alpha, 0)$ is a 2π-periodic function of α.
- Compact: Finally, we may demand that $z(\alpha, t)$ is a 2π-periodic function of α.

To obtain physically meaningful solutions in the Asymptotically Flat and Periodic flavors, we demand that

$$p(x, y, t) + gy = O(1) \quad \text{ in } \Omega(t)$$

and that

$$\int_{\Omega(t)} |v(x, y, t)|^2 dxdy < \infty \text{ (finite energy)}, \tag{1.5}$$

where we abuse of notation and identify $\Omega(t)$ with $\Omega(t) \cap \mathbb{T} \times \mathbb{R}$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, in the Periodic case.

Moreover, in Chapter 3 we study the relevance of considering the Laplace-Young condition for which the pressure on the interface $\partial\Omega(t)$ is proportional to its curvature, meaning that the surface tension effect is considered:

$$-p(z(\alpha, t), t) = \frac{\tau}{2} \frac{z_{\alpha\alpha}(\alpha, t) \cdot z_\alpha^\perp(\alpha, t)}{|z_\alpha(\alpha, t)|^3} \equiv \frac{\tau}{2} K. \tag{1.6}$$

Above $\tau > 0$ is the surface tension coefficient.

In this thesis, we restrict attention to periodic water waves, although our arguments can be easily modified to apply to the other flavors. (See Remark 2.1.5 below).

Let us summarize some of the previous work on water waves. The existence and Sobolev regularity of water waves for short time is due to S. Wu [88]. Her proof applies to smooth interfaces that need not be graphs of functions, but [88] assumes the arc-chord condition

$$|z(\alpha, t) - z(\beta, t)| \geq c_{AC}|\alpha - \beta|, \quad \text{all } \alpha, \beta \in \mathbb{R}. \tag{1.7}$$

The constant $c_{AC} > 0$ is called the arc-chord constant, which may vary with time.

The issue of long-time existence has been treated in Alvarez-Lannes [5], where well-posedness over large time scales is shown, and several asymptotic regimes are justified. By taking advantage of the dispersive properties of the water-wave system, Wu [90] proved exponentially large time of existence for small initial data.

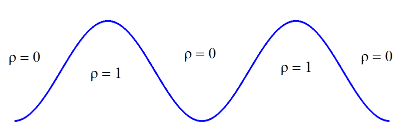
In three space dimensions, Wu [89] proved short-time existence; and Germain et al [50], [51] and Wu [91] proved existence for all time in the case of small initial data. We draw the attention of the reader to two recent preprints: the first one by Ionescu and Pusateri [63] and the second one by Alazard and Delort, [3] in which they prove existence for all time in the two-dimensional case for small initial data.

There are several important variants of the water wave problem. One can drop the assumption that the fluid is irrotational. See Christodoulou-Lindblad [26], Lindblad [70], Coutand-Shkoller [34], Shatah-Zeng [83], Zhang-Zhang [95]. Lannes [68] considered the case in which water is moving over a fixed bottom. Ambrose-Masmoudi [8] considered the case where the equations include surface tension, and the limit where the coefficient of surface tension tends to zero. Lannes [69] discussed the problem of two fluids separated by an interface with small non-zero surface tension. Alazard et al. [2] took advantage of the dispersive properties of the equations to lower the regularity of the initial data.

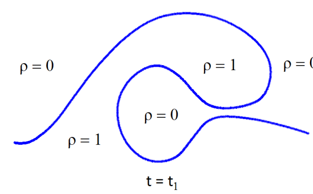
See also the papers of Córdoba et al. [28] and Alazard-Metivier [4].

In the case of large data for the two-dimensional problem (1.1-1.4), Castro et al. in [22], [21] showed that there exist initial data for which the interface is the graph of a function, but after a finite time the water wave "turns over" and the interface is no longer a graph. For previous numerical simulations showing this turning phenomenon, see Baker et al. [11] and Beale et al. [13].

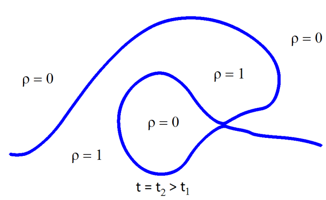
Next, we describe a singularity that can form in water waves. We start by presenting what we believe based on numerical simulations; then, we explain what we can prove.



(a) The initial water region $\Omega(t_0)$.



(b) The water region $\Omega(t_1)$ at a later time t_1.



(c) A "splash" forms at time $t_2 > t_1$.

Figure 1.1: Evolution of a "splash" singularity.

Our simulations show an initially smooth water wave, for which the fluid interface is a graph as in Figure 1.1(a). At a later time t_1, the water wave has "turned over" as described in [22], [21], i.e., the interface is no longer a graph. Finally, in Figure 1.1(c), the fluid interface self-intersects at a single point [1], but is otherwise smooth. We call this scenario a "splash",

[1]Here, we regard the fluid interface as sitting inside $\mathbb{T} \times \mathbb{R}$; recall that our water waves are 2π-periodic under horizontal translation.

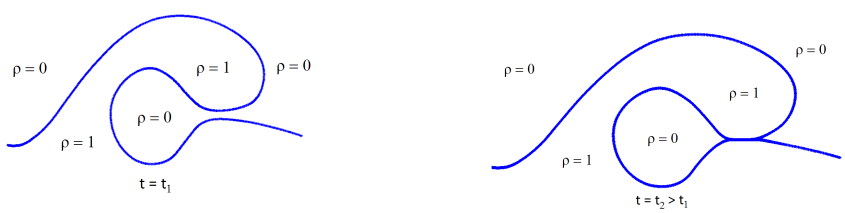
and we call the single point at which the interface self-intersects, the "splash point".

Note that the arc-chord condition holds for times $t < t_2$, but the arc-chord constant tends to zero as t tends to t_2.

Now let us explain what we can prove regarding the splash scenario. Recall that [22], [21] already proved that a water wave may start as in Figure 1.1(a) and later evolve to look like Figure 1.1(b). In this part of the thesis, we prove that a water wave may start as in Figure 1.1(b), and later form a splash, as in Figure 1.1(c).

We would like to prove that an initially smooth water wave may start as in Figure 1.1(a), then turn over as in Figure 1.1(b), and finally produce a splash as in Figure 1.1(c). To do so, our plan is to use interval arithmetic [73] to produce a rigorous computer-assisted proof that, close to the approximate solution arising from our numerics, there exists an exact solution of (1.1-1.4) that ends in a splash. The stability result announced in [19, Theorem 4.1] is a first step in this direction. A more detailed version of the plan will be given in Chapter 5.

A variant of the splash singularity is shown in Figures 1.2(a) and 1.2(b).



(a) The initial water region (b) The "splat".

Figure 1.2: Evolution of a "splat" singularity.

The water wave starts out smooth, as in Figure 1.2(a), although the interface is not a graph. At a later time, the interface self-intersects along an arc, but is otherwise smooth. In the next section, we prove that water waves can form a splat.

The stability theorem announced in [19] and proved in Chapter 5, Section 5.4 shows that a sufficiently small perturbation of the splash will come from an initial condition which is close to the one that lead to the splash. This holds true also for the splat.

We make no claim that the splash and the splat are the only singularities that can arise in solutions of the water wave equation.

Chapter 2

Splash singularity for water waves

2.1 Introduction

2.1.1 Elementary Potential Theory

To formulate precisely our main results, and to explain some ideas from their proofs, we recall some elementary potential theory for irrotational divergence-free vector fields $v(x, y, t)$ defined on a region $\Omega(t) \subset \mathbb{R}^2$ with a smooth periodic boundary $\{z(\alpha, t) : \alpha \in \mathbb{R}\}$ for fixed t. We assume that v is smooth up to the boundary and 2π-periodic with respect to horizontal translations. We suppose that v has finite energy.

Such a velocity field v may be represented in several ways:

- We may write $v = \nabla\phi$ for a velocity potential $\phi(x, y, t)$ defined on $\Omega(t)$ and smooth up to the boundary.
- We may also write $v = \nabla^{\perp}\psi = (-\partial_y\psi, \partial_x\psi)$ for a stream function ψ, defined on $\Omega(t)$ and smooth up to the boundary.
- The normal component of v at the boundary, given by

  $$u_{normal}(\alpha, t) = v(z(\alpha, t), t) \cdot \frac{(\partial_\alpha z(\alpha, t))^{\perp}}{|\partial_\alpha z(\alpha, t)|}$$

  uniquely specifies v on $\Omega(t)$. Here, $u^{\perp} = (-u_2, u_1)$ for $u = (u_1, u_2) \in \mathbb{R}^2$, and we always orient $\partial\Omega(t)$ so that the normal vector $(\partial_\alpha z(\alpha, t))^{\perp}$ points into the vacuum region $\mathbb{R}^2 \setminus \Omega(t)$.

  The function $u_{normal}(\alpha, t)$ satisfies

  $$\int_{\mathbb{T}} u_{normal}(\alpha, t)|\partial_\alpha z(\alpha, t)|d\alpha = 0,$$

  but is otherwise arbitrary.

  Note that, because v has finite energy, ϕ and ψ are 2π-periodic with respect to horizontal translations. (Without the assumption of finite energy, ϕ and ψ could be "periodic plus linear"). The functions ϕ and ψ are conjugate harmonic functions.

- There is another way to specify v, namely

$$v(x,y,t) = \frac{1}{2\pi} PV \int_{\mathbb{R}} \frac{(x - z_1(\beta,t), y - z_2(\beta,t))^{\perp}}{|(x - z_1(\beta,t), y - z_2(\beta,t))|^2} \omega(\beta,t) d\beta, \quad ((x,y) \in \Omega(t)) \tag{2.1}$$

for a 2π-periodic function $\omega(\beta,t)$ called the "vorticity amplitude". See [11].

Formula (2.1) holds only in the interior of $\Omega(t)$. Taking the limit as $(x,y) \to (z_1(\alpha,t), z_2(\alpha,t)) \in \partial\Omega(t)$ from the interior, we find that

$$v(z(\alpha,t),t) = BR(z,\omega)(\alpha,t) + \frac{1}{2}\omega(\alpha,t)\frac{\partial_\alpha z(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}, \tag{2.2}$$

where BR denotes the Birkhoff-Rott integral

$$BR(z,\omega)(\alpha,t) = \frac{1}{2\pi} P.V. \int_{\mathbb{R}} \frac{(z_1(\alpha,t) - z_1(\beta,t), z_2(\alpha,t) - z_2(\beta,t))^{\perp}}{|(z_1(\alpha,t) - z_1(\beta,t), z_2(\alpha,t) - z_2(\beta,t))|^2} \omega(\beta,t) d\beta. \tag{2.3}$$

To see that v may be represented as in (2.1), (2.2), one applies the Biot-Savart law to a discontinuous extension of v from its initial domain $\Omega(t)$ to all of $\mathbb{R}^2$; to make the extension, one solves a Neumann problem in $\mathbb{R}^2 \setminus \Omega(t)$.

Thus, our velocity field v admits multiple descriptions. Note that the description in terms of ω is significantly different from the descriptions in terms of ϕ, ψ and u_{normal}, because we bring in the Neumann problem on $\mathbb{R}^2 \setminus \Omega(t)$ to justify (2.1) and (2.2). When $\partial\Omega(t)$ is a "splash curve" as in Figure 1.1(c), there is no problem defining ϕ and it is smooth up to the boundary, except that it can take two different values at the splash point, for obvious reasons. The same is true of ψ. Similarly, $u_{normal}(\alpha,t)$ continues to behave well.

However, there is no reason to believe that $\omega(\alpha,t)$ will be well-defined and smooth for a splash curve, since $\mathbb{R}^2 \setminus \Omega(t)$ is a somewhat pathological domain. Our numerics suggest that $\max_\alpha |\omega(\alpha,t)| \sim \frac{C}{t_s - t}$, where t_s is the time of the splash.

Let us apply the above potential theory to the water wave problem. A standard formulation of the problem [11] takes $z(\alpha,t)$ and $\omega(\alpha,t)$ as unknowns. Standard computations (see e.g. [28, Section 2]) show that the water wave problem is equivalent to the following equations

$$\partial_t z(\alpha,t) = BR(z,\omega)(\alpha,t) + \overline{c}(\alpha,t)\partial_\alpha z(\alpha,t) \tag{2.4}$$

and

$$\begin{aligned} \partial_t \omega(\alpha,t) = & -2\partial_\alpha z(\alpha,t) \cdot \partial_t BR(z,\omega)(\alpha,t) \\ & - \partial_\alpha \left(\frac{|\omega|^2}{4|\partial_\alpha z|^2} \right)(\alpha,t) + \partial_\alpha \left(\overline{c}(\alpha,t)\omega(\alpha,t) \right) \\ & + 2\overline{c}(\alpha,t)\partial_\alpha z(\alpha,t) \cdot \partial_\alpha BR(z,\omega)(\alpha,t) - 2g\partial_\alpha z_2(\alpha,t). \end{aligned} \tag{2.5}$$

Here, $\overline{c}(\alpha,t)$ is a function that we may pick arbitrarily, since it influences only the parametrization of $\partial\Omega(t)$. For future reference, we write down several standard equations that follow from (1.1-1.4) by routine computation and elementary potential theory.

$$\begin{aligned}
\Delta_x\phi(x,y,t) &= \Delta_x\psi(x,y,t) = 0 \quad \text{ in } \Omega(t); \quad \phi \text{ and } \psi \text{ are harmonic conjugates.}\\
p(x,y,t) &= -\partial_t\phi(x,y,t) - \frac{1}{2}|\nabla\phi(x,y,t)|^2 - gy\\
\partial_n\psi|_{z(\alpha,t)} &= -\frac{\partial_\alpha\Phi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|}, \quad \text{ where } \Phi(\alpha,t) = \phi(z(\alpha,t),t)\\
&\quad\text{and } n \text{ is the outward-pointing unit normal to } \partial\Omega(t).\\
\psi(x+2\pi,y,t) &= \psi(x,y,t) \text{ and } \phi(x+2\pi,y,t) = \phi(x,y,t)\\
\psi(x,y,t) &= O(1) \text{ as } y \to -\infty\\
v &= \nabla^\perp\psi \text{ in } \Omega(t)\\
\partial_t z(\alpha,t) &= v(z(\alpha,t),t) + c(\alpha,t)\partial_\alpha z(\alpha,t)\\
\partial_t\Phi(\alpha,t) &= \frac{1}{2}|v(z(\alpha,t),t)|^2 + c(\alpha,t)v(z(\alpha,t),t)\cdot\partial_\alpha z(\alpha,t) - gy(\alpha,t) + p^*(t). \qquad (2.6)
\end{aligned}$$

We may write $u(\alpha,t)$ to denote $v(z(\alpha,t),t)$.

2.1.2 Main Results

Our main result is the following theorem. For the definition of a splash curve see Definition 3.3.1 in Section 2.2. The interface shown in Figure 1.1(c) is an example of a splash curve.

Theorem 2.1.1 *Let $z^0(\alpha)$ be a splash curve, where the splash point is given by $z^0(\alpha_1) = z^0(\alpha_2)$, $\alpha_1 \neq \alpha_2$. Let $u^0_{normal}(\alpha)$ be a scalar function in $H^4(\mathbb{T})$, satisfying*

$$\int_{\mathbb{T}} u^0_{normal}(\alpha)|\partial_\alpha z^0(\alpha)|d\alpha = 0 \qquad (2.7)$$

and

$$u^0_{normal}(\alpha_1), u^0_{normal}(\alpha_2) < 0. \qquad (2.8)$$

Then there exist a time $T > 0$; a time-varying domain $\Omega(t)$ defined for $t \in [0,T]$ and a velocity field $v(x,y,t)$ defined for $(x,y) \in \Omega(t)$, $t \in [0,T]$ such that the following hold:

$$\Omega(t) \text{ and } v(x,y,t) \text{ solve the water wave equations (1.1-1.4) for all } t \in [0,T]. \qquad (2.9)$$

$$\begin{aligned}&\partial\Omega(t) \text{ is given as a parametrized curve } \{z(\alpha,t) : \alpha \in \mathbb{R}\},\\ &\text{with } z(\alpha,t) - (\alpha,0)\ 2\pi\text{-periodic in } \alpha \text{ for fixed } t.\end{aligned} \qquad (2.10)$$

$$z(\alpha,t) - (\alpha,0) \in C([0,T],H^4(\mathbb{T})) \text{ and } v(z(\alpha,t),t) \in C([0,T],H^3(\mathbb{T})) \qquad (2.11)$$

$$z(\alpha,0) = z^0(\alpha) \text{ and } u_{normal}(\alpha,0) = u^0_{normal}(\alpha) \text{ for all } \alpha \in \mathbb{R}. \qquad (2.12)$$

$$\begin{aligned}&\text{For each } t \in [0,T], \text{ the curve } \partial\Omega(t) \text{ satisfies the arc-chord condition,}\\ &\text{but the arc-chord constant tends to zero as } t \to 0.\end{aligned} \qquad (2.13)$$

This result was announced in [19].

To prove that "splash singularities" can form, we note that the water wave equations are invariant under time reversal. Therefore, it is enough to exhibit a solution of the water wave equations that starts as a splash at time zero, but satisfies the arc-chord condition for each small positive time. Theorem 2.1.1 provides such solutions.

Since the curve touches itself it is not clear if the vorticity amplitude is well defined, although the velocity potential remains nonsingular. In order to get around this issue we will apply a transformation from the original coordinates to new ones which we will denote with a tilde. The purpose of this transformation is to be able to deal with the failure of the arc-chord condition. Let us consider the scenario in the periodic setting and then the transformation defined by $\tilde{z}(\alpha,t) \equiv P(z(\alpha,t))$ where P is a conformal map that will be given as:

$$P(z) = \left(\tan\left(\frac{z}{2}\right)\right)^{1/2}$$

and the branch of the root will be taken in such a way that it separates the self-intersecting points of the interface. We will also need that the interface passes below the points $(\pm\pi, 0)$ (or, equivalently, that those points belong to the vacuum region) in order for the tilde region to lie inside a closed curve and the vacuum region to lie on the outer part. See Figures 2.1 and 3.1. Here $P(z)$ will refer to a 2 dimensional vector whose components are the real and imaginary parts of $P(z_1 + iz_2)$. Its inverse is given by

$$P^{-1}(w) = i\log\left(\frac{1-iw^2}{1+iw^2}\right) = 2\arctan(w^2) \text{ for } w \in \mathbb{C}.$$

In this setting, $P^{-1}(z)$ will be well defined modulo multiples of 2π.

Remark 2.1.2 *Note that $P(z)$ is periodic such that $P(z+2k\pi) = P(z)$. Moreover, $P(z)$ is one-to-one in the water region and single-valued except at the splash point.*

Remark 2.1.3 *Although the transformation to the tilde domain is convenient, the real reason for Theorem 2.1.1 is that the potential theory inside the water region does not go bad as we approach the splash even though it goes bad in the vacuum region.*

We define the following quantities:

$$\tilde{\psi}(\tilde{x},\tilde{y},t) \equiv \psi(P^{-1}(\tilde{x},\tilde{y}),t), \quad \tilde{\phi}(\tilde{x},\tilde{y},t) \equiv \phi(P^{-1}(\tilde{x},\tilde{y}),t), \quad \tilde{v}(\tilde{x},\tilde{y},t) \equiv \nabla\tilde{\phi}(\tilde{x},\tilde{y},t),$$

$$\tilde{\Phi}(\alpha,t) = \tilde{\phi}(\tilde{z}(\alpha,t),t), \quad \tilde{\Psi}(\alpha,t) = \tilde{\psi}(\tilde{z}(\alpha,t),t).$$

Also we define $\tilde{\Omega}(t) = P(\Omega(t))$. Let us note that since ψ and ϕ are 2π periodic, the resulting $\tilde{\psi}$ and $\tilde{\phi}$ are well defined. We do not have problems with the harmonicity of $\tilde{\psi}$ or $\tilde{\phi}$ at the point which is mapped from minus infinity times i (which belongs to the water region) by P since ϕ and ψ tend to finite limits at minus infinity times i. Also, the periodicity of ϕ and ψ causes $\tilde{\phi}$ and $\tilde{\psi}$ to be continuous (and harmonic) at the interior of $P(\Omega^2(t))$.

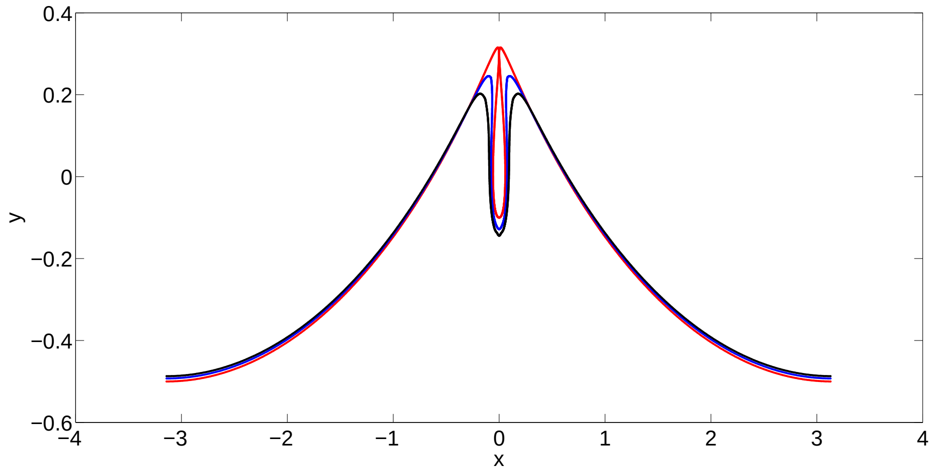


Figure 2.1: Splash singularity at times $t = 0$ (Red - splash), $t = 4 \cdot 10^{-3}$ (Blue - turning) and $t = 7 \cdot 10^{-3}$ (Black - graph).

Let us assume that there exists a solution of (2.6) and that we take $u_{normal} = \frac{\Psi_\alpha}{|z_\alpha|}$ such that $u_{normal}(\alpha_1), u_{normal}(\alpha_2) < 0$ for all $0 < t < T$, with T small enough, thus $z(\alpha, t)$ satisfies the arc-chord condition and does not touch the removed branch from $P(w)$.

The system (2.6) in the new coordinates reads

$$\begin{aligned}
\Delta\tilde{\psi}(\tilde{x}, \tilde{y}, t) &= 0 \quad \text{in } P(\Omega^2(t))\\
\partial_n \tilde{\psi}\Big|_{\tilde{z}(\alpha,t)} &= -\frac{\tilde{\Phi}_\alpha(\alpha, t)}{|\tilde{z}_\alpha(\alpha, t)|}\\
\tilde{v} &\equiv \nabla^\perp \tilde{\psi} \quad \text{in } P(\Omega^2(t))\\
\tilde{z}_t(\alpha, t) &= Q^2(\alpha, t)\tilde{u}(\alpha, t) + c(\alpha, t)\tilde{z}_\alpha(\alpha, t)\\
\tilde{\Phi}_t(\alpha, t) &= \frac{1}{2}Q^2(\alpha, t)|\tilde{u}(\alpha, t)|^2 + c(\alpha, t)\tilde{u}(\alpha, t) \cdot \tilde{z}_\alpha(\alpha, t) - gP_2^{-1}(\tilde{z}(\alpha, t))\\
\tilde{z}(\alpha, 0) &= \tilde{z}^0(\alpha)\\
\tilde{\Phi}_\alpha(\alpha, 0) &= \tilde{\Phi}^0_\alpha(\alpha) = \Phi^0_\alpha(\alpha),
\end{aligned} \tag{2.14}$$

where $\tilde{u}$ is the limit of the velocity coming from the fluid region in the tilde domain and

$$Q^2(\tilde{z}(\alpha, t), t) = \left|\frac{dP}{dw}(P^{-1}(\tilde{z}(\alpha, t)))\right|^2, \quad Q^2(\alpha, t) = \left|\frac{dP}{dw}(z(\alpha, t))\right|^2.$$

We can solve the Neumann problem in the complement of $\tilde{\Omega}(t)$. Therefore we can represent the velocity field $\tilde{v}$ in terms of a vorticity amplitude $\tilde{\omega}$.

We will see that $\tilde{z}$ and $\tilde{\omega}$ satisfy the following equations

$$\tilde{z}_t(\alpha, t) = Q^2(\alpha, t)BR(\tilde{z}, \tilde{\omega})(\alpha, t) + \tilde{c}(\alpha, t)\tilde{z}_\alpha(\alpha, t). \tag{2.15}$$

$$\begin{aligned}\tilde{\omega}_t(\alpha,t) &= -2\partial_t BR(\tilde{z},\tilde{\omega})(\alpha,t)\cdot\tilde{z}_\alpha(\alpha,t) - |BR(\tilde{z},\tilde{\omega})|^2\partial_\alpha Q^2(\alpha,t)\\ &\quad - \partial_\alpha\left(\frac{Q^2(\alpha,t)}{4}\frac{\tilde{\omega}(\alpha,t)^2}{|\tilde{z}_\alpha(\alpha,t)|^2}\right) + 2\tilde{c}(\alpha,t)\partial_\alpha BR(\tilde{z},\tilde{\omega})\cdot\tilde{z}_\alpha(\alpha,t)\\ &\quad + \partial_\alpha\left(\tilde{c}(\alpha,t)\tilde{\omega}(\alpha,t)\right) - 2g\partial_\alpha\left(P_2^{-1}(\tilde{z}(\alpha,t))\right). \qquad (2.16)\end{aligned}$$

Remark 2.1.4 *Equations (3.1-3.2) are analogous to (2.4-2.5). In fact, if we set $Q \equiv 1$ in (3.1-3.2) we recover (2.4-2.5).*

Our strategy will be the following: we will consider the evolution of the solutions in the tilde domain and then see that everything works fine in the original domain.

We will have to obtain the normal velocity once given the tangential velocity, and viceversa. To do this, we just have to notice that

$$\tilde{\Phi}_\alpha(\alpha,t) = \tilde{u}(\alpha,t)\cdot\tilde{z}_\alpha(\alpha,t) = BR(\tilde{z},\tilde{\omega})\cdot\tilde{z}_\alpha(\alpha,t) + \frac{\tilde{\omega}(\alpha,t)}{2}.$$

From that, we can invert the equation (see [28]) and get $\tilde{\omega}$. Equation (2.2) in the tilde domain then tells us $\tilde{v}$ on the boundary $\partial\tilde{\Omega}(t)$.

We now note that a solution of the system (2.14) in the tilde domain gives rise to a solution of the system (2.6) in the non-tilde domain, by inverting the map P. In fact, this will be the implication used in Theorem 2.1.1 (finding a solution in the tilde domain, and therefore in the non-tilde).

Remark 2.1.5 *It is likely that a similar argument works for the other two settings (closed contour and asymptotic to horizontal) by choosing an appropriate $P(w)$ that separates the singularity. For example, for the closed contour we could consider $P_{clo}(z) = \sqrt{z}$, taking the branch so that it separates the singularity, and for the asymptotic to horizontal scenario, it is enough to move the interface such that the water region is entirely contained in the lower halfplane (and the point $-i$ belongs to the vacuum region) and apply the relation $\frac{\tilde{z}+i}{\tilde{z}-i} = \sqrt{\frac{z+i}{z-i}}$.*

We now state the local existence results that lead to the proof of the existence of a splash singularity (Theorem 2.1.1). To avoid the failure of the arc-chord condition, we will prove the local existence in the tilde domain. This can be done in two different settings, namely in the space of analytic functions and the Sobolev space H^s.

For the analytic version we define

$$S_r = \{\alpha + i\eta, |\eta| < r\},$$

$$\|f\|^2_{L^2(\partial S_r)} = \sum_{\pm}\int_{-\pi}^{\pi}|f(\alpha \pm ir)|^2 d\alpha,$$

$$\|f\|^2_r = \|f\|^2_{L^2(\partial S_r)} + \|\partial^3_\alpha f\|^2_{L^2(\partial S_r)},$$

we consider the space

$$H^3(\partial S_r) = \left\{f \text{ analytic in } S_r, \|f\|^2_r < \infty, f\ 2\pi\text{-periodic}\right\}$$

and we take $(z_1-\alpha, z_2, \Phi) \in (H^3(\partial S_r))^3 \equiv X_r$.

The first results concerning the Cauchy problem for small data in Sobolev spaces near the equilibrium point are due to Craig [36], Nalimov [75] and Yosihara [92]. Beale et al. [14] considered the Cauchy problem in the linearized version. For local existence with small analytic data see Sulem-Sulem [85]. Our main results regarding local existence in the tilde domain are the following theorems:

Theorem 2.1.6 (Local existence for analytic initial data in the tilde domain) *Let $z^0(\alpha)$ be a splash curve and let $u^0 \cdot \frac{z^0_\alpha}{|z^0_\alpha|}(\alpha) = \frac{\Phi^0_\alpha}{|z^0_\alpha|}(\alpha)$ be the initial tangential velocity such that*

$$(z_1^0(\alpha)-\alpha, z_2^0, (\alpha), \Phi^0(\alpha)) \in X_{r_0},$$

for some $r_0 > 0$, and satisfying:

1. $u^0_{normal}(\alpha_1) = u_{normal}(\alpha_1, 0) < 0, \quad u^0_{normal}(\alpha_2) = u_{normal}(\alpha_2, 0) < 0$
2. $\displaystyle\int_{\mathbb{T}} u^0_{normal}(\alpha)|\partial_\alpha z^0(\alpha)|d\alpha = 0.$

Then there exist a finite time $T > 0$, $0 < r < r_0$, a time-varying curve $\tilde{z}(\alpha, t)$ and a function $\tilde{\Phi}(\alpha, t)$ satisfying:

1. $P^{-1}(\tilde{z}_1(\alpha, t)) - \alpha, P^{-1}(\tilde{z}_2(\alpha, t))$ *are 2π-periodic,*
2. $P^{-1}(\tilde{z}(\alpha, t))$ *satisfies the arc-chord condition for all $t \in (0, T]$,*

and $\tilde{u}(\alpha, t)$ with

$$(\tilde{z}_1(\alpha, t), \tilde{z}_2(\alpha, t), \tilde{\Phi}(\alpha, t)) \in C([0, T], X_r)$$

which provides a solution of the water wave equations (2.14) with $\tilde{z}^0(\alpha) = P(z^0(\alpha))$ and $\tilde{u}(\alpha, 0) \cdot (\tilde{z}_\alpha)^\perp(\alpha, 0) = \tilde{u}^0(\alpha) \cdot (\tilde{z}^0)^\perp_\alpha(\alpha)$.

The main tool in the proof is an abstract Cauchy-Kowalewski theorem from [78] and [79]. For more details see [24].

For the proof of local existence in Sobolev spaces we will take the following $\tilde{c}(\alpha, t)$:

$$\begin{aligned}\tilde{c}(\alpha, t) = \frac{\alpha+\pi}{2\pi}\int_{-\pi}^{\pi}(Q^2BR(\tilde{z}, \tilde{\omega}))_\beta(\beta, t)\cdot\frac{\tilde{z}_\beta(\beta, t)}{|\tilde{z}_\beta(\beta, t)|^2}d\beta \\ -\int_{-\pi}^{\alpha}(Q^2BR(\tilde{z}, \tilde{\omega}))_\beta(\beta, t)\cdot\frac{\tilde{z}_\beta(\beta, t)}{|\tilde{z}_\beta(\beta, t)|^2}d\beta.\end{aligned}$$

This choice of $\tilde{c}$ will ensure that $|\tilde{z}(\alpha, t)|$ depends only on t. We will also define an auxiliary function $\tilde{\varphi}(\alpha, t)$ analogous to the one introduced in [14] (for the linear case) and [8] (nonlinear case) which helps us to bound several of the terms that appear:

$$\tilde{\varphi}(\alpha, t) = \frac{Q^2(\alpha, t)\tilde{\omega}(\alpha, t)}{2|\tilde{z}_\alpha(\alpha, t)|} - \tilde{c}(\alpha, t)|\tilde{z}_\alpha(\alpha, t)|. \qquad (2.17)$$

Then, we can prove the following theorem:

Theorem 2.1.7 (Local existence for initial data in Sobolev spaces in the tilde domain) *In the setting of Section 2.1.1, let $\tilde{z}^0(\alpha)$ be the image of a splash curve by the map P parametrized in such a way that $|\partial_\alpha \tilde{z}^0(\alpha)|$ does not depend on α, and such that $\tilde{z}_1^0(\alpha), \tilde{z}_2^0(\alpha) \in H^4(\mathbb{T})$. Let $\tilde{\varphi}(\alpha, 0) \in H^{3+\frac{1}{2}}(\mathbb{T})$ be as in (2.17) and let $\tilde{\omega}(\alpha, 0) \in H^2(\mathbb{T})$. Then there exist a finite time $T > 0$, a time-varying curve $\tilde{z}(\alpha, t) \in C([0, T]; H^4)$, and functions $\tilde{\omega}(\alpha, t) \in C([0, T]; H^2)$ and $\tilde{\varphi} \in C([0, T]; H^{3+\frac{1}{2}})$ providing a solution of the water wave equations (3.1 - 3.2).*

The proof is based on the adaptation of the local existence proof in [28] to the tilde domain.

Some of the relevant estimates from [28] obviously hold here as well, with essentially unchanged proofs. We state such results in Lemmas 2.4.2 and Lemmas 2.4.5, ..., 2.4.9 below; and refer the reader to the relevant sections of [28] for the proofs.

However, [28] contains several "miracles", i.e., complicated calculations and estimates that lead to simple favorable results for no apparent reason. To see that analogous "miracles" occur in our present setting, we have to go through the arguments in detail; see Lemmas 2.4.10 and 2.4.12, ..., 2.4.15 below.

We have tried to make it possible to check the correctness of our arguments without extreme effort, and without undue repetitions from [28].

It would be very interesting to understand a-priori why the "miracles" in this paper and in [8], [28] occur. Presumably there is a simple, conceptual explanation, which at present we do not know.

At the end of Section 2.2 we will define the notion of a "splat curve". The curve depicted in Figure 1.2(b) is an example of a splat curve.

In the statement of Theorem 3.5.1, we may take $\tilde{z}_0(\alpha)$ to be the image of a splat curve under P rather than the image of a splash curve.

The proof of Theorem 3.5.1 goes through for this case with trivial changes. Consequently, we obtain an analogue of Theorem 2.1.6, with hypothesis 1 replaced by

Hypothesis 1′: $u^0_{normal} = u_{normal}(\alpha, 0)$ is negative for all $\alpha \in I_1 \cup I_2$, where I_1, I_2 are the intervals appearing in the definition of a splat curve in Section 2.2.

Just as Theorem 2.1.6 implies the formation of splash singularities for water waves, the above analogue of Theorem 2.1.6 for splat curves implies

Corollary 2.1.8 (Splat singularity) *There exist solutions of the water wave system that collapse along an arc in finite time, but remain otherwise smooth.*

2.1.3 Further Results

Here we mention some immediate consequences of our results which are relevant:

1. (Splash and Splat singularities for $3D$ water waves) It is possible to extend our results to the periodic three dimensional setting by considering scenarios invariant under translation in one of the coordinate directions. While preparing the final revisions of this manuscript, we noticed that in a very recent arXiv posting [35], Coutand-Shkoller consider additional 3D splash singularities.

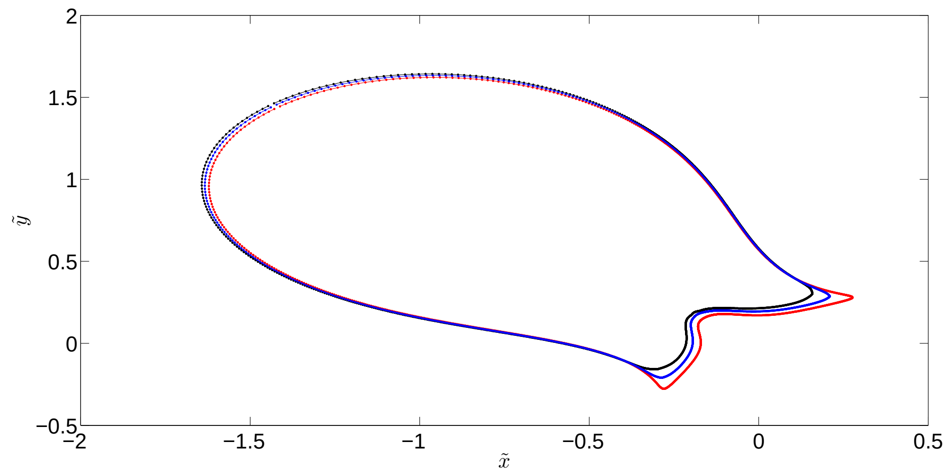


Figure 2.2: Tilde domain at times $t = 0$ (Red - splash), $t = 4 \cdot 10^{-3}$ (Blue - turning) and $t = 7 \cdot 10^{-3}$ (Black - graph).

2. (No gravity) The existence of a splash singularity can also be proved in the case where the gravity constant g is equal to zero, as long as the Rayleigh-Taylor condition holds.

2.2 Splash curves: transformation to the tilde domain and back

In this section we will rewrite the equations by applying a transformation from the original coordinates to new ones which we will denote by tilde. The purpose of this transformation is to be able to deal with the failure of the arc-chord condition.

For initial data we are interested in considering a self-intersecting curve in one point. More precisely, we will use as initial data *splash curves* which are defined this way:

Definition 2.2.1 *We say that $z(\alpha) = (z_1(\alpha), z_2(\alpha))$ is a* splash curve *if*

1. *$z_1(\alpha) - \alpha, z_2(\alpha)$ are smooth functions and 2π-periodic.*

2. *$z(\alpha)$ satisfies the arc-chord condition at every point except at α_1 and α_2, with $\alpha_1 < \alpha_2$ where $z(\alpha_1) = z(\alpha_2)$ and $|z_\alpha(\alpha_1)|, |z_\alpha(\alpha_2)| > 0$. This means $z(\alpha_1) = z(\alpha_2)$, but if we remove either a neighborhood of α_1 or a neighborhood of α_2 in parameter space, then the arc-chord condition holds.*

3. *The curve $z(\alpha)$ separates the complex plane into two regions; a connected water region and a vacuum region (not necessarily connected). The water region contains each point $x+iy$ for which y is large negative. We choose the parametrization such that the normal vector $n = \frac{(-\partial_\alpha z_2(\alpha), \partial_\alpha z_1(\alpha))}{|\partial_\alpha z(\alpha)|}$ points to the vacuum region. We regard the interface to be part of the water region.*

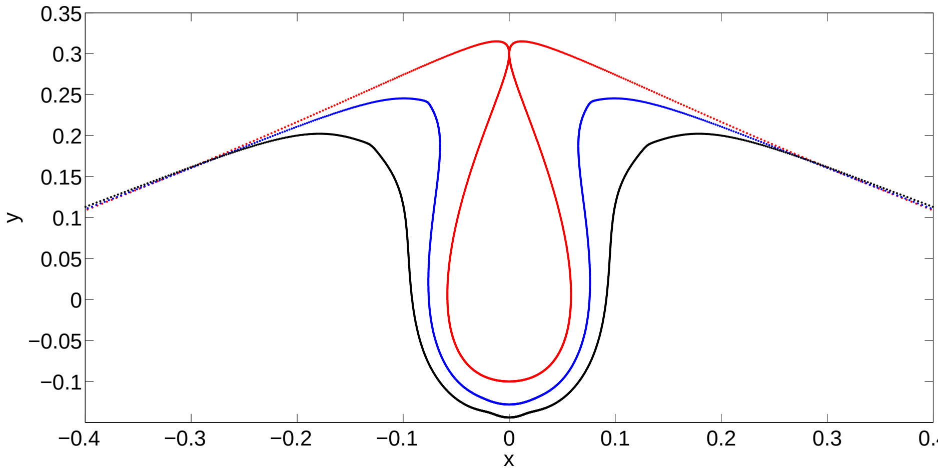


Figure 2.3: Zoom of the splash singularity at times $t = 0$ (Red - splash), $t = 4 \cdot 10^{-3}$ (Blue - turning) and $t = 7 \cdot 10^{-3}$ (Black - graph).

4. *We can choose a branch of the function P on the water region such that the curve $\tilde{z}(\alpha) = (\tilde{z}_1(\alpha), \tilde{z}_2(\alpha)) = P(z(\alpha))$ satisfies:*

   (a) *$\tilde{z}_1(\alpha)$ and $\tilde{z}_2(\alpha)$ are smooth and 2π-periodic.*

   (b) *$\tilde{z}$ is a closed contour.*

   (c) *$\tilde{z}$ satisfies the arc-chord condition.*

   We will choose the branch of the root that produces that

$$\lim_{y\to-\infty} P(x+iy) = -e^{-i\pi/4}$$

   independently of x.

5. *$P(w)$ is analytic at w and $\frac{dP}{dw}(w) \neq 0$ if w belongs to the interior of the water region. Furthermore, $(\pm\pi, 0)$ and $(0,0)$ belong to the vacuum region.*

6. *$\tilde{z}(\alpha) \neq q^l$ for $l = 0, ..., 4$, where*

$$q^0 = (0,0)\,, \quad q^1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad q^2 = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad q^3 = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right), \quad q^4 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right). \tag{2.18}$$

From now on, we will always work with splash curves as initial data unless we say otherwise. Condition 6 will be used in the local existence theorems and can be proved to hold for short enough time as long as the initial condition satisfies it. It is also immediate to check that the previous choice of P transforms any periodic interface into a closed curve. Here are two examples of curves which are not splash curves (see Figure 2.4).

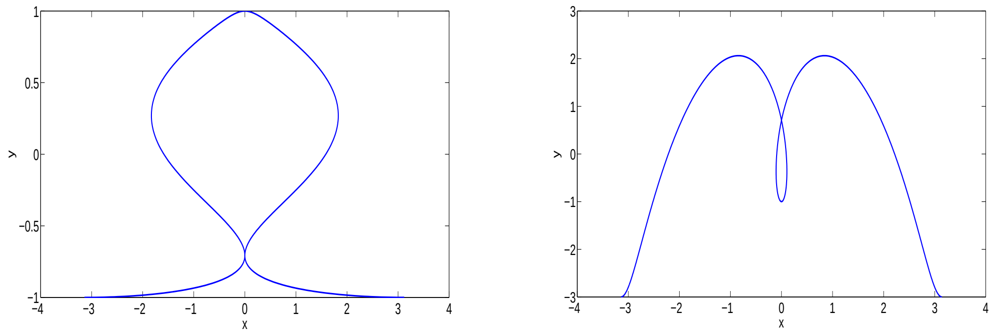


Figure 2.4: Two examples of non splash curves.

Now we will show a careful deduction of the equations in the tilde domain. From the definition of $\tilde{z}$ we have that

$$\tilde{z}_\alpha(\alpha,t) = \nabla P(z(\alpha,t)) \cdot z_\alpha(\alpha,t) \tag{2.19}$$

and

$$\begin{aligned}\tilde{z}_t(\alpha,t) &= \nabla P(z(\alpha,t)) \cdot z_t(\alpha,t) = \nabla P(z(\alpha,t)) \cdot (u(\alpha,t) + c(\alpha,t) z_\alpha(\alpha,t)) \\ &= \nabla P(z(\alpha,t)) \cdot u(\alpha,t) + c\tilde{z}_\alpha(\alpha,t).\end{aligned} \tag{2.20}$$

Since $\phi = \tilde{\phi} \circ P$ and $v = \nabla\phi = \nabla(\tilde{\phi} \circ P)$, we obtain

$$v_i = \partial_i \phi = \partial_i(\tilde{\phi} \circ P) = \sum_j (\partial_j \tilde{\phi} \circ P) \frac{\partial P_j}{\partial x_i} = \sum_j (\tilde{v}_j \circ P) \partial_i P_j. \tag{2.21}$$

This implies that

$$u(\alpha,t) = \nabla P(z(\alpha,t))^T \tilde{u}(\alpha,t). \tag{2.22}$$

Plugging this into (2.20) we get

$$\tilde{z}_t(\alpha,t) = \nabla P(z(\alpha,t)) \cdot \nabla P(z(\alpha,t))^T \cdot \tilde{u}(\alpha,t) + c\tilde{z}_\alpha(\alpha,t). \tag{2.23}$$

From the Cauchy-Riemann equations

$$\nabla P(z(\alpha,t)) \cdot \nabla P(z(\alpha,t))^T = Q^2(\alpha,t) \cdot Id_2, \quad Q^2(\alpha,t) = \left|\frac{dP(z)}{dz}\right|^2. \tag{2.24}$$

In this particular case, this means that

$$Q^2(\alpha,t) = \left|\frac{1+\tilde{z}(\alpha,t)^4}{4\tilde{z}(\alpha,t)}\right|^2, \quad \tilde{z}(\alpha,t) = \tilde{z}_1(\alpha,t) + i\tilde{z}_2(\alpha,t).$$

Recall that $\tilde{\Phi}$ is the restriction of $\tilde{\phi}$ to the interface, i.e. $\tilde{\Phi}(\alpha,t) = \tilde{\phi}(\tilde{z}(\alpha,t),t)$. Then

$$\tilde{\Phi}(\alpha,t)=\tilde{\phi}(\tilde{z}(\alpha,t),t)=\phi(P^{-1}(\tilde{z}(\alpha,t)),t)=\phi(z(\alpha,t),t)=\Phi(\alpha,t) \tag{2.25}$$

Thus, $\tilde{\Phi}$ satisfies

$$\begin{aligned}\frac{\partial\tilde{\Phi}}{\partial t}&=\frac{1}{2}|u(\alpha,t)|^2+c(\alpha,t)u(\alpha,t)\cdot z_\alpha(\alpha,t)-gz_2(\alpha,t)\\&=\frac{1}{2}|\nabla P(z(\alpha,t))^T\cdot\tilde{u}(\alpha,t)|^2+c(\alpha,t)\tilde{u}(\alpha,t)\cdot\tilde{z}_\alpha(\alpha,t)-gP_2^{-1}(\tilde{z}(\alpha,t)),\end{aligned} \tag{2.26}$$

where the subscript in the gravity term of the last line denotes the second component. Thus the system (2.6) in the new coordinates reads

$$\begin{aligned}\Delta\tilde{\psi}(x,y,t)&=0\quad\text{in } P(\Omega^2(t))\\ \partial_n\tilde{\psi}\Big|_{\tilde{z}(\alpha,t)}&=-\frac{\tilde{\Phi}_\alpha(\alpha,t)}{|\tilde{z}_\alpha(\alpha,t)|}\\ \tilde{v}&\equiv\nabla^\perp\tilde{\psi}\quad\text{in } P(\Omega^2(t))\\ \tilde{z}_t(\alpha,t)&=Q^2(\alpha,t)\tilde{u}(\alpha,t)+c(\alpha,t)\tilde{z}_\alpha(\alpha,t)\\ \tilde{\Phi}_t(\alpha,t)&=\frac{1}{2}Q^2(\alpha,t)|\tilde{u}(\alpha,t)|^2+c(\alpha,t)\tilde{u}(\alpha,t)\cdot\tilde{z}_\alpha(\alpha,t)-gP_2^{-1}(\tilde{z}(\alpha,t))+p^*(t)\\ \tilde{z}(\alpha,0)&=\tilde{z}^0(\alpha)\\ \tilde{\Phi}_\alpha(\alpha,0)&=\tilde{\Phi}^0_\alpha(\alpha)=\Phi^0_\alpha(\alpha).\end{aligned} \tag{2.27}$$

We have seen that $\tilde{v}$ can be represented in the form

$$\tilde{v}(\tilde{x},\tilde{y},t)=\nabla^\perp\tilde{\psi}(\tilde{x},\tilde{y},t)=\frac{1}{2\pi}P.V\int_{-\pi}^{\pi}\frac{(\tilde{x}-\tilde{z}_1(\alpha,t),\tilde{y}-\tilde{z}_2(\alpha,t))^\perp}{|(\tilde{x}-\tilde{z}_1(\alpha,t),\tilde{y}-\tilde{z}_2(\alpha,t))|^2}\tilde{\omega}(\alpha,t)d\alpha.$$

Taking limits from the fluid region we obtain

$$\tilde{u}(\alpha,t)=BR(\tilde{z},\tilde{\omega})+\frac{\tilde{\omega}}{2|\tilde{z}_\alpha|^2}\tilde{z}_\alpha.$$

The evolution of $\tilde{\omega}$ is calculated in the following way. First, let us recall the equations

$$\begin{aligned}\tilde{z}_t(\alpha,t)&=Q^2(\alpha,t)\tilde{u}(\alpha,t)+c(\alpha,t)\tilde{z}_\alpha(\alpha,t)\\ \tilde{\Phi}_t(\alpha,t)&=\frac{1}{2}Q^2(\alpha,t)|\tilde{u}(\alpha,t)|^2+c(\alpha,t)\tilde{u}(\alpha,t)\cdot\tilde{z}_\alpha(\alpha,t)-gP_2^{-1}(\tilde{z}(\alpha,t))\\ \tilde{\Phi}_\alpha(\alpha,t)&=\tilde{u}(\alpha,t)\cdot\tilde{z}_\alpha(\alpha,t)\\ \tilde{z}(\alpha,0)&=\tilde{z}^0(\alpha)\\ \tilde{\Phi}_\alpha(\alpha,0)&=\tilde{\Phi}^0_\alpha(\alpha)=\Phi^0_\alpha(\alpha).\end{aligned} \tag{2.28}$$

Substituting the expression for $\tilde{u}(\alpha,t)$ and performing the change $\tilde{c}(\alpha,t)=c(\alpha,t)+\frac{1}{2}Q^2(\alpha,t)\frac{\tilde{\omega}(\alpha,t)}{|\tilde{z}_\alpha(\alpha,t)|^2}$ we obtain

$$
\begin{aligned}
\tilde{z}_t(\alpha,t) &= Q^2(\alpha,t)BR(\tilde{z},\tilde{\omega})(\alpha,t) + \tilde{c}(\alpha,t)\tilde{z}_\alpha(\alpha,t)\\
\tilde{\Phi}_\alpha(\alpha,t) &= BR(\tilde{z},\tilde{\omega})(\alpha,t)\cdot\tilde{z}_\alpha(\alpha,t) + \frac{1}{2}\tilde{\omega}(\alpha,t)\\
\tilde{\Phi}_t(\alpha,t) &= \frac{1}{2}Q^2(\alpha,t)|\tilde{u}(\alpha,t)|^2 + c(\alpha,t)\tilde{u}(\alpha,t)\cdot\tilde{z}_\alpha(\alpha,t) - gP_2^{-1}(\tilde{z}(\alpha,t))\\
&= \frac{1}{2}Q^2(\alpha,t)|BR(\tilde{z},\tilde{\omega})(\alpha,t)|^2 - \frac{Q^2(\alpha,t)}{8}\frac{\tilde{\omega}(\alpha,t)^2}{|\tilde{z}_\alpha(\alpha,t)|^2}\\
&+ \tilde{c}(\alpha,t)BR(\tilde{z},\tilde{\omega})\cdot\tilde{z}_\alpha(\alpha,t) + \frac{1}{2}\tilde{c}(\alpha,t)\tilde{\omega}(\alpha,t) - gP_2^{-1}(\tilde{z}(\alpha,t)).
\end{aligned}\tag{2.29}
$$

On the one hand, by taking derivatives with respect to t in the second equation follows

$$
\begin{aligned}
\tilde{\Phi}_{\alpha t}(\alpha,t) &= \partial_t BR(\tilde{z},\tilde{\omega})(\alpha,t)\cdot\tilde{z}_\alpha(\alpha,t) + BR(\tilde{z},\tilde{\omega})(\alpha,t)\cdot\tilde{z}_{\alpha t}(\alpha,t) + \frac{\tilde{\omega}_t(\alpha,t)}{2}\\
&= \partial_t BR(\tilde{z},\tilde{\omega})\cdot\tilde{z}_\alpha(\alpha,t) + |BR(\tilde{z},\tilde{\omega})|^2\partial_\alpha Q^2(\alpha,t)\\
&+ Q^2(\alpha,t)BR(\tilde{z},\tilde{\omega})\cdot\partial_\alpha BR(\tilde{z},\tilde{\omega}) + \tilde{c}_\alpha(\alpha,t)BR(\tilde{z},\tilde{\omega})\cdot\tilde{z}_\alpha(\alpha,t)\\
&+ \tilde{c}(\alpha,t)BR(\tilde{z},\tilde{\omega})\cdot\tilde{z}_{\alpha\alpha}(\alpha,t) + \frac{\tilde{\omega}_t(\alpha,t)}{2}.
\end{aligned}\tag{2.30}
$$

On the other, taking derivatives with respect to α in the third equation in (2.29) yields

$$
\begin{aligned}
\tilde{\Phi}_{\alpha t}(\alpha,t) &= \frac{1}{2}|BR(\tilde{z},\tilde{\omega})|^2\partial_\alpha Q^2(\alpha,t) + Q^2(\alpha,t)BR(\tilde{z},\tilde{\omega})\cdot\partial_\alpha BR(\tilde{z},\tilde{\omega})\\
&- \frac{1}{2}\partial_\alpha\left(\frac{Q^2(\alpha,t)}{4}\frac{\tilde{\omega}(\alpha,t)^2}{|\tilde{z}_\alpha(\alpha,t)|^2}\right) + \tilde{c}_\alpha(\alpha,t)BR(\tilde{z},\tilde{\omega})\cdot\tilde{z}_\alpha(\alpha,t)\\
&+ \tilde{c}(\alpha,t)\partial_\alpha BR(\tilde{z},\tilde{\omega})\cdot\tilde{z}_\alpha(\alpha,t) + \tilde{c}(\alpha,t)BR(\tilde{z},\tilde{\omega})\cdot\tilde{z}_{\alpha\alpha}(\alpha,t)\\
&+ \frac{1}{2}\partial_\alpha\left(\tilde{c}(\alpha,t)\tilde{\omega}(\alpha,t)\right) - \partial_\alpha\left(gP_2^{-1}(\tilde{z}(\alpha,t))\right).
\end{aligned}\tag{2.31}
$$

Combining both equations, we find that

$$
\begin{aligned}
\tilde{\omega}_t(\alpha,t) &= -2\partial_t BR(\tilde{z},\tilde{\omega})(\alpha,t)\cdot\tilde{z}_\alpha(\alpha,t) - |BR(\tilde{z},\tilde{\omega})|^2\partial_\alpha Q^2(\alpha,t)\\
&- \partial_\alpha\left(\frac{Q^2(\alpha,t)}{4}\frac{\tilde{\omega}(\alpha,t)^2}{|\tilde{z}_\alpha(\alpha,t)|^2}\right) + 2\tilde{c}(\alpha,t)\partial_\alpha BR(\tilde{z},\tilde{\omega})\cdot\tilde{z}_\alpha(\alpha,t)\\
&+ \partial_\alpha\left(\tilde{c}(\alpha,t)\tilde{\omega}(\alpha,t)\right) - 2\partial_\alpha\left(gP_2^{-1}(\tilde{z}(\alpha,t))\right).
\end{aligned}\tag{2.32}
$$

We will proceed in the following way: we will consider the evolution of the solutions in the tilde domain and see that everything works fine in the original domain. For example, the sign condition on the normal vectors in the non-tilde domain has an equivalent form in the tilde domain (i.e. the two normal components have negative sign).

In the non-tilde domain, this implies that the interface moves away from the branch removed from the square root, and therefore the interface touches neither the branch cut nor the conflictive points q^l (see Condition 6 in Definition 3.3.1). Hence P and P^{-1} will be well defined and one-to-one. (See Figure 2.7).

Let us note that getting $\phi = \tilde{\phi} \circ P$ is not a problem since ϕ is bounded and harmonic. Moreover, as $\tilde{v} = \nabla^{\perp}\tilde{\psi}$ and

$$v = \nabla P^T(\tilde{v} \circ P)$$

and ∇P has exponential decay at infinity, the velocity v belongs to $L^2(\Omega^2(t) \cap [-\pi, \pi] \times \mathbb{R})$.

Remark 2.2.2 *Ψ, Φ, u and z have easy transformations to the tilde domain but ω has not.*

We would like to discuss what happens to the amplitude of the vorticity ω in the non-tilde domain as the curve approaches the splash.

If the vorticity belongs to $C([0, T_{splash}], C^{\delta}(\mathbb{T}))$, then the normal velocity should be continuous at the splash point and therefore the normal component of the restriction of the velocity to the curve from the water region cannot have the same sign at $z(\alpha_1)$ and $z(\alpha_2)$ (see Theorem 2.1.1). This means that the $C^{\delta}-$norm of the amplitude of the vorticity becomes unbounded at the time of the splash.

We illustrate this phenomenon by plotting $1/\max|\omega|$ (see Figure 2.5), where the blue curve is the calculated ω and the red curve is a potential fitting to the data as numerical instabilities don't allow us to compute ω with enough precision when we are in the regime which is close to the splash. Time has been reversed so that the splash occurs at time $t = 0$ and the interface separates from itself at $t > 0$.

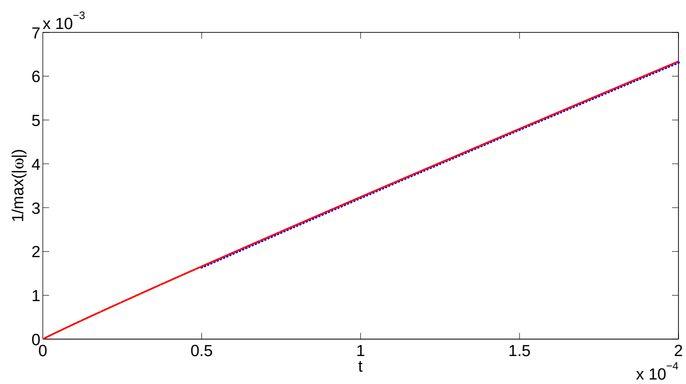


Figure 2.5: Vorticity amplitude in the nontilde domain. The vorticity reaches infinity at a rate of approximately $\frac{1}{(T_{splash}-t)^{0.966}} \approx \frac{1}{(T_{splash}-t)}$. The fit is given by $F = 23.72 \cdot t^{0.966} - 1.476 \cdot 10^{-6}$.

We also have performed numerical simulations in order to get a blowup rate for the arc-chord condition. As in Figure 2.5, we plot the inverse of the arc-chord constant. The blue curve is made by the calculated points and the red curve is the interpolating one. We see a very good fitting. Time follows the same convention as before and the numerical evidence indicates a blowup of the arc-chord as $\frac{1}{T_{splash}-t}$. The results can be seen in Figure 2.6.

The numerics that led us to Figures 1.1(a), 1.1(b) and 1.1(c) were performed using the method of Beale-Hou-Lowengrub [14], with special modifications to maintain accuracy up to the splash (i.e. taking into account the impact of Q on the equation). The code was written first in Matlab, and then ported into C++ (GSL) [49] to optimize in terms of speed. We

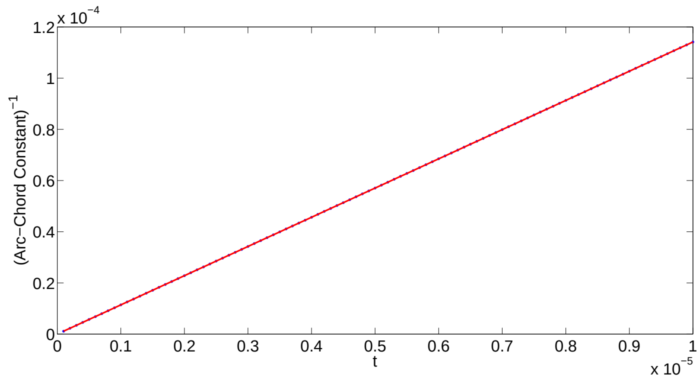


Figure 2.6: Arc-chord condition in the non-tilde domain. The arc-chord reaches infinity at a rate of approximately $\frac{1}{(T_{splash}-t)}$. The fit is given by $F = 11.41 \cdot t + 5.104 \cdot 10^{-9}$.

enclose the Matlab code in Appendix for clarity reasons. Actual results from our simulations are shown in Figures 2.1, 3.1 and 2.3. Figures 1.1 and 1.2 are cartoons.

Instead of having an evolution equation for ω, Beale-Hou-Lowengrub introduce a velocity potential ϕ and study its evolution through time subject to the constraint imposed by being a potential. This is the set of equations (2.28). The initial data on the non-tilde domain was given by:

$$z_1^0(\alpha) = \alpha + \frac{1}{4}\left(-\frac{3\pi}{2} - 1.9\right)\sin(\alpha) + \frac{1}{2}\sin(2\alpha) + \frac{1}{4}\left(\frac{\pi}{2} - 1.9\right)\sin(3\alpha)$$
$$z_2^0(\alpha) = \frac{1}{10}\cos(\alpha) - \frac{3}{10}\cos(2\alpha) + \frac{1}{10}\cos(3\alpha)$$

Note that $z\left(\frac{\pi}{2}\right) = z\left(-\frac{\pi}{2}\right)$ (splash). Instead of prescribing an initial condition for ω, we prescribed the normal component of the velocity to ensure a more controlled direction of the fluid. From that we got the initial $\omega(\alpha, 0)$ using the following relations. Let ψ be such that $\nabla^{\perp}\psi = v$ and $\Psi(\alpha)$ its restriction to the interface. Recall that we can transform the initial condition on the normal component of the velocity into an initial condition on the tangential component by applying the transformations described in section 2. The initial normal velocity is then prescribed by setting

$$u_n^0(\alpha)|z_\alpha(\alpha)| = \Psi_\alpha(\alpha) = 3 \cdot \cos(\alpha) - 3.4 \cdot \cos(2\alpha) + \cos(3\alpha) + 0.2\cos(4\alpha).$$

The simulations were done using a spatial mesh of $N = 2048$ nodes and a time step $\Delta t = 10^{-7}$. The time direction was set to run backwards (from the splash to the graph) and the graph was obtained at approximately $T_g = 6.5 \cdot 10^{-3}$.

We also kept track of the energy conservation. If we consider the following energy (not to be confused with the one in Section 2.4):

$$E_S(t) = \frac{1}{2}\int_{\Omega_f^2(t)} |v(x,y,t)|^2 dxdy + \frac{1}{2}\int_{-\pi}^{\pi} g(z_2(\alpha,t))^2 \partial_\alpha z_1(\alpha,t)d\alpha \equiv E_k(t) + E_p(t) \quad (2.33)$$

where $z(\alpha,t)=(z_1(\alpha,t),z_2(\alpha,t)), u(\alpha,t)=v(z(\alpha,t),t)$, and $\Omega^2_f(t)=\Omega^2(t)\cap[-\pi,\pi]\times\mathbb{R}$ is a fundamental domain in the water region in a period, then we can see that the energy is conserved; this is a check of the accuracy of our numerics.

$$\begin{aligned}\frac{dE_k(t)}{dt} &= \int_{\Omega^2_f(t)} v(x,y,t)(v_t(x,y,t)+v(x,y,t)\cdot\nabla v(x,y,t))dxdy\\ &= \int_{\Omega^2_f(t)} v(x,y,t)(-\nabla p(x,y,t)-g(0,1))dxdy\\ &= -\int_{\Omega^2_f(t)} v(x,y,t)(\nabla(p(x,y,t)+gy))dxdy\\ &= -\int_{\partial(\Omega^2_f(t))} v(x,y,t)\cdot\overrightarrow{n}gyds\\ &= -\int_{-\pi}^{\pi} gz_2(\alpha,t)u(\alpha,t)\cdot\partial_\alpha z^\perp(\alpha,t)d\alpha \end{aligned} \tag{2.34}$$

where we have used the incompressibility of the fluid ($\nabla\cdot v=0$) and the continuity of the pressure on the interface ($p^*(t)|_{\partial\Omega^2(t)}=0$). Next

$$\begin{aligned}\frac{dE_p(t)}{dt} &= \int_{-\pi}^{\pi} gz_2(\alpha,t)\partial_t z_2(\alpha,t)\partial_\alpha z_1(\alpha,t)d\alpha+\frac{1}{2}\int_{-\pi}^{\pi} g(z_2(\alpha,t))^2\partial_t\partial_\alpha z_1(\alpha,t)d\alpha\\ &= \int_{-\pi}^{\pi} gz_2(\alpha,t)\partial_t z_2(\alpha,t)\partial_\alpha z_1(\alpha,t)d\alpha-\int_{-\pi}^{\pi} gz_2(\alpha,t)\partial_\alpha z_2(\alpha,t)\partial_t z_1(\alpha,t)d\alpha\\ &= \int_{-\pi}^{\pi} gz_2(\alpha,t)u(\alpha,t)\cdot\partial_\alpha z^\perp(\alpha,t)d\alpha. \end{aligned} \tag{2.35}$$

This proves that the energy is constant. Note that:

$$\begin{aligned}\int_{\Omega^2_f(t)}|v(x,y,t)|^2dxdy &= \int_{\Omega^2_f(t)}|\nabla\phi(x,y,t)|^2dxdy\\ &= -\int_{\Omega^2_f(t)}\phi(x,y,t)\Delta\phi(x,y,t)dxdy\\ &\quad+\int_{\partial(\Omega^2_f(t))}\phi(x,y,t)\nabla\phi(x,y,t)\cdot\overrightarrow{n}dxdy\\ &\overset{\Delta\phi=0}{=}\int_{\partial(\Omega^2_f(t))}\phi(x,y,t)\nabla\phi(x,y,t)\cdot\overrightarrow{n}dxdy \end{aligned} \tag{2.36}$$

so the numerical calculation is restricted to the values at the boundary. We observe that the energy of our system is conserved, as we have

$$E_S(t)\approx 38.3936,\qquad \frac{\max_t E_S(t)-\min_t E_S(t)}{\min_t E_S(t)}\approx 6\cdot 10^{-11}.$$

We now give the proof of Theorem 2.1.1 using Theorem 3.5.1.

Proof of Theorem 2.1.1: Using the fact that there is local existence to the initial data in the tilde domain and applying P^{-1} to the solution obtained there, we can get a curve $z(\alpha,t)$ that solves the water wave equation in the non tilde domain. Details on the local existence in the tilde domain are shown below. Note that the sign condition (2.8) assumed in Theorem 2.1.1 guarantees that for positive time t the curve in the nontilde domain will separate (as depicted in Figure 2.7(a)) instead of crossing itself (as depicted in Figure 2.7(b)). More precisely, we check that for small positive time t the curve $\alpha \mapsto z(\alpha,t) = (z_1(\alpha,t), z_2(\alpha,t)) = P^{-1}(\tilde{z}(\alpha,t)) \in \mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}$ is a simple closed curve, i.e. that $\alpha \mapsto z(\alpha,t)$ is one-to-one. Indeed, if not, there exist a sequence of positive times $t_\nu \to 0$ and points $\alpha_\nu^{'}, \alpha_\nu^{''}$ such that $\alpha_\nu^{'} \neq \alpha_\nu^{''}$ mod $2\pi\mathbb{Z}$, but $z(\alpha_\nu^{'}, t_\nu) = z(\alpha_\nu^{''}, t_\nu)$. Since the initial splash curve $\alpha \mapsto z(\alpha,0)$ satisfies the modified chord-arc condition described in Condition 2 of Definition 3.3.1, we may assume without loss of generality that $\alpha_\nu^{'} \to \alpha_1$ and $\alpha_\nu^{''} \to \alpha_2$ (with α_1, α_2 as in Definition 3.3.1). The sign condition (2.8) therefore guarantees that (for large ν), $\tilde{z}(\alpha_\nu^{'}, t_\nu)$ and $\tilde{z}(\alpha_\nu^{''}, t_\nu)$ lie in the image of the (open) time-zero water region under the map P. Moreover (for large ν), $\tilde{z}(\alpha_\nu^{'}, t_\nu) \neq \tilde{z}(\alpha_\nu^{''}, t_\nu)$ since $\tilde{z}(\alpha_1, 0) \neq \tilde{z}(\alpha_2, 0)$.

Since P^{-1} is one-to-one on the image of the open time-zero water region under P, it follows that (for large ν) we have $z(\alpha_\nu^{'}, t_\nu) \neq z(\alpha_\nu^{''}, t_\nu) \in \mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}$, with $z(\alpha,t) \equiv P^{-1}(\tilde{z}(\alpha,t))$. This contradicts the defining condition $z(\alpha_\nu^{'}, t_\nu) = z(\alpha_\nu^{''}, t_\nu)$, completing the proof that $\alpha \mapsto z(\alpha,t)$ is a simple closed curve for small positive t.

The proof of Theorem 2.1.1 is complete. □

We end this section by defining a "splat curve", as promised in Section 2.1. To do so, we simply modify our Definition 3.3.1 for a splash curve, by replacing Condition 2 in that definition by the following

Condition 2′: We are given two disjoint closed non-degenerate intervals $I_1, I_2 \subset [0, 2\pi)$ whose images under $\alpha \mapsto (z_1(\alpha), z_2(\alpha)) \in \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$ coincide.

The map $\alpha \mapsto (z_1(\alpha), z_2(\alpha)) \in \mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}$ satisfies the chord-arc condition when restricted to the complement of any open interval J such that $J \supset I_1$ or $J \supset I_2$.

As promised, the curve depicted in Figure 1.2(b) is a splat curve. Observe that the curve in Figure 1.2(b) cannot be real-analytic.

2.3 Proof of real-analytic short-time existence in tilde domain

The main goal of this section is to prove Theorem 2.1.6. In order to accomplish this task we will prove local well-posedness for the system (2.37) below. In this section, we will drop the tildes from the notation. The system arises from (2.28) taking $c = 0$:

$$\begin{cases} z_t &= \left|\frac{dP}{dw}(P^{-1}(z))\right|^2 u \\ \Phi_t &= \frac{1}{2}\left|\frac{dP}{dw}(P^{-1}(z))\right|^2 |u|^2 - gP_2^{-1}(z) \\ u &= BR(z,\omega) + \frac{\omega}{2|z_\alpha|^2} z_\alpha \\ \Phi_\alpha &= \frac{\omega}{2} + BR(z,\omega)\cdot z_\alpha \\ \left|\frac{dP}{dw}(P^{-1}(z(\alpha,t)))\right|^2 &= \frac{1}{16}\left|\frac{1+(z_1(\alpha,t)+iz_2(\alpha,t))^4}{z_1(\alpha,t)+iz_2(\alpha,t)}\right|^2 \\ P_2^{-1}(z(\alpha,t)) &= \log\left|\frac{i+(z_1(\alpha,t)+iz_2(\alpha,t))^2}{i-(z_1(\alpha,t)+iz_2(\alpha,t))^2}\right|. \end{cases} \tag{2.37}$$

We demand that $z^0(\alpha) \neq (0,0)$ to find the function $\frac{dP}{dw}(P^{-1}(z(\alpha,t)))$ well defined. This condition is going to remain true for short time. We also consider $z^0(\alpha) \neq q_l$, $l = 1, ..., 4$ in (3.8) to get $P_2^{-1}(z(\alpha,t))$ well defined. Again this is going to remain true for short time.

The main tool in this section is a Cauchy-Kowalewski theorem (see [22, Section 5] for more details). We recall the following definitions

$$S_r = \{\alpha + i\eta, |\eta| < r\},$$

$$\|f\|^2_{L^2(\partial S_r)} = \sum_{\pm} \int_{-\pi}^{\pi} |f(\alpha \pm ir)|^2 d\alpha,$$

$$\|f\|^2_r = \|f\|^2_{L^2(\partial S_r)} + \|\partial^3_\alpha f\|^2_{L^2(\partial S_r)},$$

the space

$$H^3(\partial S_r) = \big\{f \text{ analytic in } S_r, \|f\|^2_r < \infty, f\ 2\pi\text{-periodic}\big\}$$

and we now take $(z_1, z_2, \Phi) \in (H^3(\partial S_r))^3 \equiv X_r$. We have the following theorem:

Theorem 2.3.1 *Let $z^0(\alpha)$ be a curve satisfying the arc-chord condition*

$$\frac{|z^0(\alpha) - z^0(\alpha-\beta)|^2}{|\beta|^2} > \frac{1}{M^2}$$

which doesn't touch the points q_l, $l = 0, ..., 4$ in (3.8),and $(z^0, \Phi^0) \in X_{r_0}$ for some $r_0 > 0$. Then, there exist a time $T > 0$ and $0 < r < r_0$ such that there is a unique solution to the system (2.37) in $C([0,T], X_r)$ with initial conditions $z(\alpha,0) = z^0(\alpha), \Phi(\alpha,0) = \Phi^0(\alpha)$, for all $\alpha \in \mathbb{T}$.

Equation (2.37) can be extended for complex variables:

$$z_t(\alpha+i\xi,t) = F^1(z(\alpha+i\xi,t), \Phi(\alpha+i\xi,t)), \quad \Phi_t(\alpha+i\xi,t) = F^2(z(\alpha+i\xi,t), \Phi(\alpha+i\xi,t)).$$

Here

$$F^1(z,\Phi) = \left|\frac{dP}{dw}(P^{-1}(z))\right|^2 u$$

where we abuse notation by writing

$$\left|\frac{dP}{dw}(P^{-1}(z(\alpha+i\xi,t)))\right|^2 = \frac{1}{16}\frac{\Pi_{l=1}^4[(z_1(\alpha+i\xi,t) - q_1^l)^2 + (z_2(\alpha+i\xi,t) - q_2^l)^2]}{(z_1(\alpha+i\xi,t))^2 + (z_2(\alpha+i\xi,t))^2}$$

and

$$u(\alpha+i\xi,t) = BR(z(\alpha+i\xi,t), \omega(\alpha+i\xi,t)) + \frac{1}{2}\left(\frac{\omega(\alpha+i\xi,t)\partial_\alpha z(\alpha+i\xi,t)}{(\partial_\alpha z_1(\alpha+i\xi,t))^2 + (\partial_\alpha z_2(\alpha+i\xi,t))^2}\right)$$

with

$$BR(z(\alpha+i\xi,t), \omega(\alpha+i\xi,t)) =$$

$$\frac{1}{2\pi}PV\int_{\mathbb{T}}\frac{(z_2(\alpha+i\xi-\beta,t)-z_2(\alpha+i\xi,t),z_1(\alpha+i\xi,t)-z_1(\alpha+i\xi-\beta,t))}{(z_1(\alpha+i\xi,t)-z_1(\alpha+i\xi-\beta,t))^2+(z_2(\alpha+i\xi,t)-z_2(\alpha+i\xi-\beta,t))^2}\omega(\alpha+i\xi-\beta,t)d\beta$$

and ω given implicitly by

$$\Phi_\alpha = \frac{\omega}{2} + BR(z,\omega)\cdot z_\alpha.$$

We will also abuse notation by writing $|u|^2$ for $u_1^2+u_2^2$, even for complex $u=(u_1,u_2)$. The operator F^2 is given by

$$F^2(z,\Phi) = \frac{1}{2}\left|\frac{dP}{dw}\left(P^{-1}(z)\right)\right|^2|u|^2 - gP_2^{-1}(z)$$

where

$$P_2^{-1}(z(\alpha+i\xi,t)) = \frac{1}{2}\sum_{l=1}^{4}(-1)^l\log[(z_1(\alpha+i\xi,t)-q_1^l)^2+(z_2(\alpha+i\xi,t)-q_2^l)^2].$$

Below we will use a strip of analyticity small enough so that the complex logarithm above is continuous. We use the following proposition:

Proposition 2.3.2 *Consider $0\le r<r'$ and the open set $O\subset X_{r'}$ given by:*

$$O=\{(z,\Phi)\in X_{r'}:\|z_i\|_{r'},\|\Phi\|_{r'}<R,\inf_{\alpha+i\xi\in S_r}|(z_1(\alpha+i\xi)-q_1^l)^2+(z_2(\alpha+i\xi)-q_2^l)^2|>R^{-2},$$
$$l=0,...,4,\ \inf_{\substack{\alpha+i\xi\in S_r\\ \beta\in[-\pi,\pi]}}G(z)(\alpha+i\xi,\beta)>R^{-2}\}$$

with

$$G(z)(\alpha+i\xi,\beta)=\left|\frac{(z_1(\alpha+i\xi)-z_1(\alpha+i\xi-\beta))^2+(z_2(\alpha+i\xi)-z_2(\alpha+i\xi-\beta))^2}{\beta^2}\right|$$

then the function $F=(F^1,F^2)$ for $F:O\to X_r$ is a continuous mapping. In addition, there is a constant C_R (depending on R only) such that

$$\|F(z,\Phi)\|_r\le\frac{C_R}{r'-r}\|(z,\Phi)\|_{r'} \tag{2.38}$$

$$\|F(z^2,\Phi^2)-F(z^1,\Phi^1)\|_r\le\frac{C_R}{r'-r}\|(z^2-z^1,\Phi^2-\Phi^1)\|_{r'} \tag{2.39}$$

and

$$\sup_{\substack{\alpha+i\xi\in S_r\\ \beta\in[-\pi,\pi]}}|F^1(z,\Phi)(\alpha+i\xi)-F^1(z,\Phi)(\alpha+i\xi-\beta)|\le C_R|\beta| \tag{2.40}$$

for $z,z^j,\Phi,\Phi^j\in O$.

Proof: First we point out that ω is given in term of Φ_α and z by the implicit equation

$$\Phi_\alpha = \frac{\omega}{2} + BR(z,\omega)\cdot z_\alpha \equiv \frac{1}{2}(I+J)(\omega).$$

It is well known that the operator $(I+J)$ is invertible on L^2 for real functions with mean zero (see [28, Section 5] for more details). Writing

$$\omega(\alpha \pm ir) = 2\Phi_\alpha(\alpha \pm ir) - \frac{1}{2\pi}\int_{\mathbb{T}} \frac{(z(\alpha \pm ir) - z(\beta))^\perp \cdot z_\alpha(\alpha \pm ir)}{|z(\alpha \pm ir) - z(\beta)|^2}\omega(\beta)d\beta$$

one can find that

$$\|\omega\|_{L^2(\partial S_r)} \le 2\|\Phi_\alpha\|_{L^2(\partial S_r)} + C_R\|\omega\|_{L^2(\partial S_0)}$$

(where C_R depends on R) for $(z,\Phi)\in O$. The bound of $(I+J)^{-1}$ for real functions yields

$$\|\omega\|_{L^2(\partial S_0)} \le 2\|(I+J)^{-1}\|_{L^2\to L^2}\|\Phi_\alpha\|_{L^2(\partial S_0)} \le C_R\|\Phi_\alpha\|_{L^2(\partial S_r)}.$$

Thus

$$\|\omega\|_{L^2(\partial S_r)} \le C_R\|\Phi_\alpha\|_{L^2(\partial S_r)}.$$

Analogously, one finds that

$$\|\partial_\alpha^2\omega\|_{L^2(\partial S_r)} \le C_R\|\Phi\|_r.$$

This allows us to assert that ω is at the same level as Φ_α in terms of derivatives:

$$\|\omega\|_{L^2(\partial S_r)} + \|\partial_\alpha^2\omega\|_{L^2(\partial S_r)} \le C_R\|\Phi\|_r \le C_R\|\Phi\|_{r'}. \tag{2.41}$$

Then, inequality (2.38) follows as in [22, Section 6.3]. We will see how to deal with the most singular terms. For the first term in the norm, it is easy to find that

$$\|F(z,\Phi)\|_{L^2(\partial S_r)} \le C_R\|(z,\Phi)\|_r \le C_R\|(z,\Phi)\|_{r'}. \tag{2.42}$$

In order to control the second one, we will show how to deal with F^1 as F^2 is analogous. Here we point out that the functions

$$\left|\frac{dP}{dw}(P^{-1}(z(\alpha+i\xi,t)))\right|^2, \quad P_2^{-1}(z(\alpha+i\xi,t))$$

have no loss of derivatives and they are regular as long as $(z,\Phi)\in O$. Therefore, in $\partial_\alpha^3 F^1$ the most singular term is given by

$$\left|\frac{dP}{dw}(P^{-1}(z(\alpha+i\xi,t)))\right|^2 \partial_\alpha^3 u(\alpha+i\xi,t)$$

as the rest can be estimated in an easier manner (see [28, Section 6.1] as an example with more details). From the definition it is easy to bound $\left|\frac{dP}{dw}(P^{-1}(z))\right|^2$ in L^∞, it remains to control $\partial_\alpha^3 u$ in $L^2(\partial S_r)$. To simplify the exposition we ignore the time dependence of the functions, we denote $\gamma = \alpha \pm ir$,

$$(z_1(\gamma) - z_1(\gamma-\beta))^2 + (z_2(\gamma) - z_2(\gamma-\beta))^2 \equiv |z(\gamma) - z(\gamma-\beta)|_*^2,$$

$$(\partial_\alpha z_1(\gamma))^2 + (\partial_\alpha z_2(\gamma))^2 \equiv |z_\alpha(\gamma)|_*^2,$$

and

$$(z_2(\gamma-\beta) - z_2(\gamma), z_1(\gamma) - z_1(\gamma-\beta)) \equiv (z(\gamma) - z(\gamma-\beta))^\perp.$$

Next, we split as follows

$$\partial_\alpha^3 u = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + \text{l.o.t.}$$

where l.o.t. denotes lower order terms which can be estimated in an easier manner. We have

$$I_1 = \frac{1}{2\pi} PV \int_{-\pi}^{\pi} \frac{(\partial_\alpha^3 z(\gamma) - \partial_\alpha^3 z(\gamma-\beta))^\perp}{|z(\gamma) - z(\gamma-\beta)|_*^2} \omega(\gamma-\beta) d\beta,$$

$$I_2 = \frac{-1}{\pi} PV \int_{-\pi}^{\pi} \frac{(z(\gamma)-z(\gamma-\beta))^\perp}{(|z(\gamma)-z(\gamma-\beta)|_*^2)^2} (z(\gamma)-z(\gamma-\beta)) \cdot (\partial_\alpha^3 z(\gamma)-\partial_\alpha^3 z(\gamma-\beta))\omega(\gamma-\beta) d\beta,$$

$$I_3 = \frac{1}{2\pi} PV \int_{-\pi}^{\pi} \frac{(z(\gamma) - z(\gamma-\beta))^\perp}{|z(\gamma) - z(\gamma-\beta)|_*^2} \partial_\alpha^3 \omega(\gamma-\beta) d\beta,$$

$$I_4 = \frac{1}{2} \frac{\omega(\gamma)\partial_\alpha^4 z(\gamma)}{|z_\alpha(\gamma)|_*^2},$$

$$I_5 = -\frac{1}{2} \frac{\omega(\gamma)\partial_\alpha z(\gamma)}{(|z_\alpha(\gamma)|_*^2)^2} \partial_\alpha z(\gamma) \cdot \partial_\alpha^4 z(\gamma)$$

and

$$I_6 = \frac{1}{2} \frac{\partial_\alpha^3 \omega(\gamma) \partial_\alpha z(\gamma)}{|z_\alpha(\gamma)|_*^2}.$$

For I_6 we find

$$\|I_6\|_{L^2(\partial S_r)} \le \frac{1}{2} \|\partial_\alpha z\|_{L^\infty(S_r)} \Big(\inf_{\substack{\gamma \in S_r \\ \beta \in [-\pi,\pi]}} G(z)(\gamma,\beta) \Big)^{-1} \|\partial_\alpha^3 \omega\|_{L^2(\partial S_r)}$$

and since $(z, \Phi) \in O$ we get

$$\|I_6\|_{L^2(\partial S_r)} \le C_R \|\partial_\alpha^3 \omega\|_{L^2(\partial S_r)}$$

by using Sobolev embedding. A simple application of the Cauchy formula gives

$$\|\partial_\alpha f\|_{L^2(\partial S_r)} \le \frac{C}{r' - r} \|f\|_{L^2(\partial S_{r'})}$$

which allows us to find

$$\|I_6\|_{L^2(\partial S_r)} \le \frac{C_R}{r' - r} \|\partial_\alpha^2 \omega\|_{L^2(\partial S_{r'})}.$$

The bound (2.41) gives finally

$$\|I_6\|_{L^2(\partial S_r)} \le \frac{C_R}{r' - r} \|\Phi\|_{r'}.$$

In a similar way we obtain

$$\|I_4\|_{L^2(\partial S_r)} + \|I_5\|_{L^2(\partial S_r)} \le C_R \|\partial_\alpha^4 z\|_{L^2(\partial S_r)} \le \frac{C_R}{r'-r}\|z\|_{r'}.$$

In I_3 we decompose further: $I_3 = I_{3,1} + I_{3,2}$ where

$$I_{3,1} = \frac{1}{2\pi} PV \int_{-\pi}^{\pi} K(\gamma,\beta)\partial_\alpha^3\omega(\gamma-\beta)d\beta, \quad I_{3,2} = \frac{1}{2}\frac{z_\alpha^\perp(\gamma)}{|z_\alpha(\gamma)|_*^2} H(\partial_\alpha^3\omega)(\gamma),$$

where H denotes the Hilbert transform and the kernel K is given by

$$\frac{(z(\gamma)-z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|_*^2} - \frac{z_\alpha^\perp(\gamma)}{|z_\alpha(\gamma)|_*^2}\frac{1}{2\tan(\beta/2)}.$$

We can integrate by parts $\partial_\beta(-\partial_\alpha^2\omega(\gamma-\beta))$ in $I_{3,1}$ to find

$$\|I_{3,1}\|_{L^2(\partial S_r)} \le C_R \|\partial_\alpha^2\omega\|_{L^2(\partial S_r)} \le C_R\|\Phi\|_{r'}$$

(see [28, Section 3] for more details). The term $I_{3,2}$ can be estimated by

$$\|I_{3,2}\|_{L^2(\partial S_r)} \le C_R\|H(\partial_\alpha^3\omega)\|_{L^2(\partial S_r)} = C_R\|\partial_\alpha^3\omega\|_{L^2(\partial S_r)} \le \frac{C_R}{r'-r}\|\Phi\|_{r'}.$$

A similar splitting in $I_2 = I_{2,1} + I_{2,2}$ with

$$I_{2,1} = \frac{-1}{\pi} PV \int_{-\pi}^{\pi} L(\gamma,\beta)\cdot(\partial_\alpha^3 z(\gamma) - \partial_\alpha^3 z(\gamma-\beta))d\beta,$$

$$I_{2,2} = -\frac{\omega(\gamma)z_\alpha^\perp(\gamma)}{(|z_\alpha(\gamma)|_*^2)^2} z_\alpha(\gamma)\cdot\Lambda(\partial_\alpha^3 z)(\gamma),$$

(where $\Lambda = H\partial_\alpha$) gives the kernel L as follows

$$L(\gamma,\beta)\cdot(\partial_\alpha^3 z(\gamma) - \partial_\alpha^3 z(\gamma-\beta)) = -\frac{\omega(\gamma)z_\alpha^\perp(\gamma)}{(|z_\alpha(\gamma)|_*^2)^2}\frac{z_\alpha(\gamma)\cdot(\partial_\alpha^3 z(\gamma)-\partial_\alpha^3 z(\gamma-\beta))}{4\sin^2(\beta/2)}$$
$$+\frac{\omega(\gamma-\beta)(z(\gamma)-z(\gamma-\beta))^\perp}{(|z(\gamma)-z(\gamma-\beta)|_*^2)^2}(z(\gamma)-z(\gamma-\beta))\cdot(\partial_\alpha^3 z(\gamma)-\partial_\alpha^3 z(\gamma-\beta)).$$

Heuristically, we regard this operator as no better or no worse than a Hilbert transform of $\partial_\alpha^3 z$. It is easy to prove that

$$\|I_{2,1}\|_{L^2(\partial S_r)} \le C_R\|\partial_\alpha^3 z\|_{L^2(\partial S_r)} \le C_R\|\Phi\|_{r'}$$

(see [28, Section 6.1] for more details). The term $I_{2,2}$ can be bounded as follows

$$\|I_{2,2}\|_{L^2(\partial S_r)} \le C_R\|\Lambda(\partial_\alpha^3 z)\|_{L^2(\partial S_r)} = C_R\|\partial_\alpha^4 z\|_{L^2(\partial S_r)} \le \frac{C_R}{r'-r}\|z\|_{r'}.$$

Analogously, for I_1 we find

$$\|I_1\|_{L^2(\partial S_r)} \leq \frac{C_R}{r'-r}\|z\|_{r'}.$$

This strategy allows us to deal with $\partial_\alpha^3 u$ and therefore with $\partial_\alpha^3 F^1$. The same applies to $\partial_\alpha^3 F^2$ and we can get finally (2.38).

To get (2.39) we write

$$\Phi_\alpha^1 = \frac{1}{2}(I + J_{z^1})(\omega^1), \quad \Phi_\alpha^2 = \frac{1}{2}(I + J_{z^2})(\omega^2)$$

where

$$J_{z^j}(\omega) = 2BR(z^j, \omega) \cdot z_\alpha^j$$

for $z^j \in O$ and $j = 1, 2$. This implies

$$\Phi_\alpha^2 - \Phi_\alpha^1 = \frac{\omega^2 - \omega^1}{2} + BR(z^2, \omega^2 - \omega^1) \cdot z_\alpha^2 + BR(z^2, \omega^1) \cdot z_\alpha^2 - BR(z^1, \omega^1) \cdot z_\alpha^1$$

which yields

$$(\omega^2 - \omega^1) = 2(I + J_{z^2})^{-1}(\Phi_\alpha^2 - \Phi_\alpha^1) - 2(I + J_{z^2})^{-1}(BR(z^2, \omega^1) \cdot z_\alpha^2 - BR(z^1, \omega^1) \cdot z_\alpha^1).$$

This helps us to find

$$\|\omega^2 - \omega^1\|_{L^2(\partial S_r)} + \|\partial_\alpha^2 \omega^2 - \partial_\alpha^2 \omega^1\|_{L^2(\partial S_r)} \leq C(R)(\|\Phi^2 - \Phi^1\|_r + \|z^2 - z^1\|_r).$$

We use a decomposition similar to the one used to prove (2.38) which allows us to get finally (2.39). Inequality (2.40) follows in an easier manner. □

Proof of Theorem 2.3.1: We apply the following result of Nirenberg [78] and Nishida [79].

Abstract Cauchy-Kowalewski Theorem:
Consider the equation

$$\frac{du(t)}{dt} = F(u(t)) \text{ for } |t| < \delta \tag{2.43}$$

with initial condition

$$u(0) = u^0 \in X_{r_0} \tag{2.44}$$

For some numbers $\hat{C}, \hat{R} > 0$, assume the following hypothesis:

For every pair of numbers r, r' such that $0 < r' < r < r_0$, F is a Lipschitz map from $\{u \in X_r : \|u - u^0\|_{X_r} < \hat{R}\}$ into $X_{r'}$, with Lipschitz constant at most $\dfrac{\hat{C}}{r - r'}$. Then the equation (2.43) with initial condition (2.44) has a solution $u(t)$ in $C([-\delta, \delta], X_r)$ for small enough $r, \delta > 0$.

The above Abstract Cauchy-Kowalewski Theorem is obviously equivalent to a special case of Nishida's Theorem [79], although our notation differs from that of [78]. In place of (2.43), Nirenberg and Nishida treat the more general equation $\dfrac{du(t)}{dt} = F(u(t), t)$.

The proof of the Abstract Cauchy-Kowalewski Theorem in [78] proceeds by showing that the obvious iteration scheme

$$u^{k+1}(t) = u^0 + \int_0^t F(u^k(s))ds$$

converges in X_r for small enough r (depending on t).

Our system (2.37) has the form $\frac{du}{dt} = F(u)$ for $u = (z, \Phi)$. Proposition 2.3.2 tells us that the hypothesis of the Abstract Cauchy-Kowalewski Theorem holds for the system (2.37). In particular, for $\hat{R} > 0$ small enough, we obtain the arc-chord condition for every $u = (z, \Phi)$ such that $\|(z, \Phi) - (z^0, \Phi^0)\|_{X_r} < \hat{R}$ for any (arbitrarily small) $r > 0$.

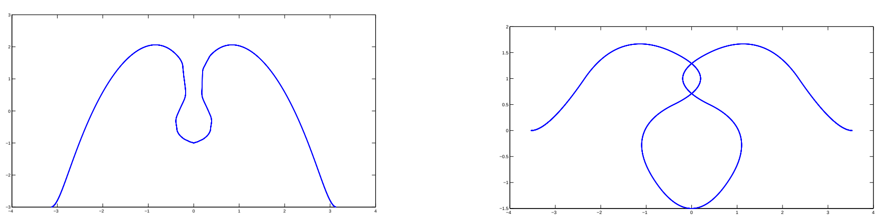
Hence, the conclusion of Theorem 2.3.1 follows from the Abstract Cauchy-Kowalewski Theorem.

$\square$

Proof of Theorem 2.1.6:

Applying Theorem 2.3.1, we obtain a solution of the water wave equation, with the correct initial conditions, in the tilde domain. Passing from the tilde domain back to the original problem, we obtain a solution of the water wave equations as asserted in Theorem 2.1.6.

We have to make sure that, for small positive time, the splash curve evolves as in Figure 2.7(a), rather than Figure 2.7(b).



(a) Good

(b) Bad

Figure 2.7: Two different evolutions of the interface.

This is guaranteed by the hypothesis of Theorem 2.1.6 regarding the sign of the normal component of the initial velocity at the splash point.

$\square$

2.4 Proof of short-time existence in Sobolev spaces in the tilde domain

In this section we will show how to obtain a local existence theorem for the water wave equations in the tilde domain. The proof is based on energy estimates and uses the fact that the Rayleigh-Taylor function is positive.

2.4.1 The Rayleigh-Taylor function in the tilde domain

We begin by recalling the function $\tilde{\varphi}(\alpha,t)$, which will be studied in detail in Section 2.4.3 and in the definition of the Rayleigh-Taylor condition, by the expression

$$\tilde{\varphi}(\alpha,t)=\frac{Q^2\tilde{\omega}(\alpha,t)}{2|\tilde{z}_\alpha(\alpha,t)|}-\tilde{c}|\tilde{z}_\alpha(\alpha,t)|. \tag{2.45}$$

Next we introduce the R-T function:

$$\begin{aligned}\sigma\equiv&\left(BR_t(\tilde{z},\tilde{\omega})+\frac{\tilde{\varphi}}{|\tilde{z}_\alpha|}BR_\alpha(\tilde{z},\tilde{\omega})\right)\cdot\tilde{z}_\alpha^\perp+\frac{\tilde{\omega}}{2|\tilde{z}_\alpha|^2}\left(\tilde{z}_{\alpha t}+\frac{\tilde{\varphi}}{|\tilde{z}_\alpha|}\tilde{z}_{\alpha\alpha}\right)\cdot\tilde{z}_\alpha^\perp\\&+Q\left|BR(\tilde{z},\tilde{\omega})+\frac{\tilde{\omega}}{2|\tilde{z}_\alpha|^2}\tilde{z}_\alpha\right|^2(\nabla Q)(\tilde{z})\cdot\tilde{z}_\alpha^\perp+g(\nabla P_2^{-1})(\tilde{z})\cdot\tilde{z}_\alpha^\perp.\end{aligned} \tag{2.46}$$

This function σ coincides with the expression $\tilde{z}^\perp(\alpha,t)\cdot\nabla\tilde{p}(\tilde{z}(\alpha,t),t)$, where $\tilde{p}=p\circ P^{-1}$. Indeed, it is easy to check that

$$\partial_t\tilde{\phi}+\frac{Q^2}{2}|\tilde{v}|^2=-\tilde{p}-gP_2^{-1}+p^*(t). \tag{2.47}$$

And taking the gradient on the equation (2.47) yields

$$\tilde{v}_t+\frac{1}{2}\left(\nabla Q^2\right)|\tilde{v}|^2+Q^2(\tilde{v}\cdot\nabla)\tilde{v}=-\nabla\tilde{p}-g\nabla P_2^{-1}. \tag{2.48}$$

In addition we know that

$$\tilde{v}(\tilde{z}(\alpha,t),t)=BR(\tilde{z},\tilde{\omega})(\alpha,t)+\frac{\tilde{\omega}(\alpha,t)}{2|\tilde{z}_\alpha(\alpha,t)|^2}\tilde{z}_\alpha(\alpha,t) \tag{2.49}$$

and therefore

$$\frac{d}{dt}\tilde{v}(\tilde{z}(\alpha,t),t)=\partial_t BR(\tilde{z},\tilde{\omega})(\alpha,t)+\partial_t\left(\frac{\tilde{\omega}(\alpha,t)}{2|\tilde{z}_\alpha(\alpha,t)|^2}\right)\tilde{z}_\alpha(\alpha,t)+\frac{\tilde{\omega}(\alpha,t)}{2|\tilde{z}_\alpha(\alpha,t)|^2}\partial_t\tilde{z}_\alpha(\alpha,t). \tag{2.50}$$

On the other hand, by using (2.48) we have

$$\begin{aligned}\frac{d}{dt}\tilde{v}(\tilde{z}(\alpha,t),t)=&\partial_t\tilde{v}(\tilde{z}(\alpha,t),t)+(\partial_t\tilde{z}(\alpha,t)\cdot\nabla)\tilde{v}(\tilde{z}(\alpha,t),t)\\=&-\frac{1}{2}\left(\nabla Q^2\right)|\tilde{v}(\tilde{z}(\alpha,t),t)|^2-Q^2(\tilde{v}(\tilde{z}(\alpha,t),t)\cdot\nabla)\tilde{v}(\tilde{z}(\alpha,t),t)\\&-\nabla\tilde{p}(\tilde{z}(\alpha,t),t)-g\nabla P_2^{-1}(\tilde{z}(\alpha,t))+(\partial_t\tilde{z}(\alpha,t)\cdot\nabla)\tilde{v}(\tilde{z}(\alpha,t),t).\end{aligned} \tag{2.51}$$

Furthermore the equation (2.29) together with (2.49) gives rise to

$$\begin{aligned}\partial_t\tilde{z}(\alpha,t)=&Q^2\tilde{v}(\tilde{z}(\alpha,t),t)-\frac{Q^2\tilde{\omega}(\alpha,t)}{2|\tilde{z}_\alpha(\alpha,t)|^2}\tilde{z}_\alpha(\alpha,t)+\tilde{c}\tilde{z}_\alpha(\alpha,t)\\=&Q^2\tilde{v}(\tilde{z}(\alpha,t),t)-\frac{1}{|\tilde{z}_\alpha(\alpha,t)|}\left(\frac{Q^2\tilde{\omega}(\alpha,t)}{2|\tilde{z}_\alpha(\alpha,t)|}-\tilde{c}|\tilde{z}_\alpha(\alpha,t)|\right)\tilde{z}_\alpha(\alpha,t).\end{aligned} \tag{2.52}$$

Therefore by (2.45), we obtain

$$\partial_t \tilde{z}(\alpha,t) = Q^2 \tilde{v}(\tilde{z}(\alpha,t),t) - \tilde{\varphi}(\alpha,t)\frac{\tilde{z}_\alpha(\alpha,t)}{|\tilde{z}_\alpha(\alpha,t)|}. \tag{2.53}$$

By introducing (2.53) in (2.51) we have

$$\begin{aligned}\frac{d}{dt}\tilde{v}(\tilde{z}(\alpha,t),t) =& -\frac{1}{2}\left(\nabla Q^2\right)|\tilde{v}(\tilde{z}(\alpha,t),t)|^2 - Q^2(\tilde{v}(\tilde{z}(\alpha,t),t)\cdot\nabla)\tilde{v}(\tilde{z}(\alpha,t),t)\\ &-\nabla\tilde{p}(\tilde{z}(\alpha,t),t) - g\nabla P_2^{-1}(\tilde{z}(\alpha,t))\\ &+Q^2(\tilde{v}(\tilde{z}(\alpha,t),t)\cdot\nabla)\tilde{v}(\tilde{z}(\alpha,t),t) - \tilde{\varphi}(\alpha,t)\frac{\tilde{z}_\alpha(\alpha,t)}{|\tilde{z}_\alpha(\alpha,t)|}\cdot\nabla\tilde{v}(\tilde{z}(\alpha,t),t).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{d}{dt}\tilde{v}(\tilde{z}(\alpha,t),t) =& -\frac{1}{2}\left(\nabla Q^2\right)|\tilde{v}(\tilde{z}(\alpha,t),t)|^2 - \tilde{\varphi}(\alpha,t)\frac{\partial_\alpha\tilde{v}(\tilde{z}(\alpha,t),t)}{|\tilde{z}_\alpha(\alpha,t)|}\\ &-\nabla\tilde{p}(\tilde{z}(\alpha,t),t) - g\nabla P_2^{-1}(\tilde{z}(\alpha,t)).\end{aligned} \tag{2.54}$$

Next we take a derivative with respect to α in the equation (2.49) to get

$$\partial_\alpha\tilde{v}(\tilde{z}(\alpha,t),t) = \partial_\alpha BR(\tilde{z},\tilde{\omega})(\alpha,t) + \partial_\alpha\left(\frac{\tilde{\omega}(\alpha,t)}{2|\tilde{z}_\alpha(\alpha,t)|^2}\right)\tilde{z}_\alpha(\alpha,t) + \left(\frac{\tilde{\omega}(\alpha,t)}{2|\tilde{z}_\alpha(\alpha,t)|^2}\right)\tilde{z}_{\alpha\,\alpha}(\alpha,t). \tag{2.55}$$

Multiplying equation (2.54) by $\tilde{z}_\alpha^\perp(\alpha,t)$ and using (2.55) we learn

$$\begin{aligned}\left(\frac{d}{dt}\tilde{v}(\tilde{z}(\alpha,t),t)\right)\cdot\tilde{z}_\alpha^\perp(\alpha,t) =& -Q\nabla Q\cdot\tilde{z}_\alpha^\perp(\alpha,t)|\tilde{v}(\tilde{z}(\alpha,t),t)|^2\\ &-\frac{\tilde{\varphi}(\alpha,t)}{|\tilde{z}_\alpha(\alpha,t)|}\partial_\alpha BR(\tilde{z},\tilde{\omega})(\alpha,t)\cdot\tilde{z}_\alpha^\perp(\alpha,t)\\ &-\frac{\tilde{\varphi}(\alpha,t)}{|\tilde{z}_\alpha(\alpha,t)|}\left(\frac{\tilde{\omega}(\alpha,t)}{2|\tilde{z}_\alpha(\alpha,t)|^2}\right)\tilde{z}_{\alpha\,\alpha}(\alpha,t)\cdot\tilde{z}_\alpha^\perp(\alpha,t)\\ &-\nabla\tilde{p}(\tilde{z}(\alpha,t),t)\cdot\tilde{z}_\alpha^\perp(\alpha,t) - g\nabla P_2^{-1}(\tilde{z}(\alpha,t))\cdot\tilde{z}_\alpha^\perp(\alpha,t).\end{aligned} \tag{2.56}$$

On the other hand, by multiplying (2.50) by $\tilde{z}_\alpha^\perp(\alpha,t)$ we have

$$\left(\frac{d}{dt}\tilde{v}(\tilde{z}(\alpha,t),t)\right)\cdot\tilde{z}_\alpha^\perp(\alpha,t) = \partial_t BR(\tilde{z},\omega)\cdot\tilde{z}_\alpha^\perp(\alpha,t) + \frac{\tilde{\omega}}{2|\tilde{z}_\alpha(\alpha,t)|^2}\partial_t\tilde{z}_\alpha(\alpha,t)\cdot\tilde{z}_\alpha^\perp(\alpha,t). \tag{2.57}$$

From (2.56) and (2.57) we find

$$\begin{aligned}&\partial_t BR(\tilde{z},\omega)\cdot\tilde{z}_\alpha^\perp(\alpha,t) + \frac{\tilde{\omega}}{2|\tilde{z}_\alpha(\alpha,t)|^2}\partial_t\tilde{z}_\alpha(\alpha,t)\cdot\tilde{z}_\alpha^\perp(\alpha,t)\\ =& -Q\nabla Q\cdot\tilde{z}_\alpha^\perp(\alpha,t)|\tilde{v}(\tilde{z}(\alpha,t),t)|^2 - \frac{\tilde{\varphi}(\alpha,t)}{|\tilde{z}_\alpha(\alpha,t)|}\partial_\alpha BR(\tilde{z},\tilde{\omega})(\alpha,t)\cdot\tilde{z}_\alpha^\perp(\alpha,t)\\ &-\frac{\tilde{\varphi}(\alpha,t)}{|\tilde{z}_\alpha(\alpha,t)|}\left(\frac{\tilde{\omega}(\alpha,t)}{2|\tilde{z}_\alpha(\alpha,t)|^2}\right)\tilde{z}_{\alpha\,\alpha}(\alpha,t)\cdot\tilde{z}_\alpha^\perp(\alpha,t)\\ &-\nabla\tilde{p}(\tilde{z}(\alpha,t),t)\cdot\tilde{z}_\alpha^\perp(\alpha,t) - g\nabla P_2^{-1}(\tilde{z}(\alpha,t))\cdot\tilde{z}_\alpha^\perp(\alpha,t).\end{aligned} \tag{2.58}$$

Finally, rearranging the terms in (2.58) yields

$$\begin{aligned}
&-\nabla\tilde{p}(\tilde{z}(\alpha,t),t)\cdot\tilde{z}_\alpha^\perp(\alpha,t) = \left(\partial_t BR(\tilde{z},\tilde{\omega})(\alpha,t) + \frac{\tilde{\varphi}(\alpha,t)}{|\tilde{z}_\alpha(\alpha,t)|}\partial_\alpha BR(\tilde{z},\tilde{\omega})(\alpha,t)\right)\cdot\tilde{z}_\alpha^\perp(\alpha,t)\\
&+\frac{\tilde{\omega}(\alpha,t)}{2|\tilde{z}_\alpha(\alpha,t)|^2}\left(\partial_t\tilde{z}_\alpha(\alpha,t) + \frac{\tilde{\varphi}(\alpha,t)}{|\tilde{z}_\alpha(\alpha,t)|}\tilde{z}_{\alpha\,\alpha}\right)\cdot\tilde{z}_\alpha^\perp(\alpha,t) + g\nabla P_2^{-1}\cdot\tilde{z}_\alpha^\perp(\alpha,t)\\
&+Q\left|BR(\tilde{z},\tilde{\omega})(\alpha,t) + \frac{\tilde{\omega}(\alpha,t)}{2|\tilde{z}_\alpha(\alpha,t)|^2}\tilde{z}_\alpha(\alpha,t)\right|^2\left(\nabla Q\cdot\tilde{z}_\alpha^\perp(\alpha,t)\right),
\end{aligned}$$

and then, comparing with (5.2), we obtain the desired result

$$-\nabla\tilde{p}(\tilde{z}(\alpha,t),t)\cdot\tilde{z}_\alpha^\perp(\alpha,t) = \sigma(\alpha,t).$$

Note that for the tilde domain, the Rayleigh-Taylor condition is the same as in the first domain, i.e:

$$\nabla p(\alpha,t)\cdot z_\alpha^\perp(\alpha,t) = \nabla\tilde{p}(\alpha,t)\cdot\tilde{z}_\alpha^\perp(\alpha,t)$$

where $\tilde{p} = p\circ P^{-1}$ and

$$\tilde{z}_\alpha(\alpha,t) = \nabla P(z(\alpha,t))\cdot z_\alpha(\alpha,t) \Rightarrow \tilde{z}_\alpha^\perp(\alpha,t) = (-J\nabla P(z(\alpha,t))J)\cdot z_\alpha^\perp(\alpha,t)$$

where J is the rotation matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Together with the Cauchy-Riemann equations this implies that

$$(-J\nabla P(z(\alpha,t))J) = \nabla P(z(\alpha,t)).$$

Moreover

$$\nabla p(\alpha,t) = \nabla P(z(\alpha,t))^T\nabla\tilde{p}(\alpha,t).$$

Hence

$$\begin{aligned}
\langle\nabla p(\alpha,t), z_\alpha^\perp(\alpha,t)\rangle &= \langle\nabla P(z(\alpha,t))^T\nabla\tilde{p}(\alpha,t), (\nabla P(z(\alpha,t)))^{-1}\tilde{z}_\alpha^\perp(\alpha,t)\rangle && (2.59)\\
&= \langle\nabla\tilde{p}(\alpha,t), \tilde{z}_\alpha^\perp(\alpha,t)\rangle. && (2.60)
\end{aligned}$$

By taking the divergence on the Euler equation (1.1-1.2) and because the flow is irrotational in the interior of the regions $\Omega^j(t)$ follows

$$-\Delta p = |\nabla v|^2 \geq 0$$

which, together with the fact that the pressure is zero on the interface and $p(x,y,t)+gy = O(1)$ when y tends to $-\infty$,then follows by Hopf's lemma in $\Omega^2(t)$ that

$$\sigma(\alpha,t) \equiv -|z_\alpha^\perp(\alpha,t)|\partial_n p(z(\alpha,t),t) > 0,$$

except in the case $v = 0$. This argument was suggested by Hou and Caflisch (see [89]), although the proof of the positivity of the Rayleigh-Taylor condition in the nontilde domain for all time was first introduced by Wu in [88].

The above proof shows that $\sigma > 0$ provided our domain $\tilde{\Omega}(t)$ arises by applying the map P to a domain $\Omega(t)$ with smooth boundary. Here, $\partial\Omega(t)$ may be a splash curve, but we cannot allow boundaries $\partial\tilde{\Omega}(t)$ whose inverse images under P look like figure 2.7(b).

Nevertheless, since $\sigma > 0$ for the image of P applied to a splash curve, we know that $\sigma > 0$ at time $t = 0$ in the context of Theorem 3.5.1. Our estimates below will guarantee that the condition $\sigma > 0$ persists for a short time. Thus, in proving Theorem 3.5.1, we may use the positivity of σ.

2.4.2 Definition of c in the tilde domain

From now on, we will drop the tildes from the notation for simplicity. We will choose the following tangential term:

$$c = \frac{\alpha+\pi}{2\pi}\int_{-\pi}^{\pi}(Q^2BR)_\beta\cdot\frac{z_\beta}{|z_\beta|^2}d\beta - \int_{-\pi}^{\alpha}(Q^2BR)_\beta\cdot\frac{z_\beta}{|z_\beta|^2}d\beta. \tag{2.61}$$

Here and in (2.29) we find

$$P_2^{-1} = P_2^{-1}(z(\alpha,t)) = \log\left(\left|\frac{i+(z_1(\alpha,t)+iz_2(\alpha,t))^2}{i-(z_1(\alpha,t)+iz_2(\alpha,t))^2}\right|\right)$$

and

$$Q = Q(z(\alpha,t)) = \frac{1}{4}\left|\frac{1+(z_1(\alpha,t)+iz_2(\alpha,t))^4}{z_1(\alpha,t)+iz_2(\alpha,t)}\right|.$$

These functions are regular as long as $z(\alpha,t) \neq q^l$. We deal with initial data which satisfy the above condition and we will show that it's going to remain true for short time. In order to measure it we define

$$m(q^l)(t) = \min_{\alpha\in\mathbb{T}}|z(\alpha,t)-q^l|$$

for $l = 0, ..., 4$.

We also point out that, because of our choice of $c(\alpha,t)$, solutions of (3.1 - 3.2) satisfy that

$$|z_\alpha(\alpha,t)|^2 = A(t) \quad \text{for any } \alpha\in\mathbb{T}$$

as in [29, Equations (2.2 - 2.5)].

2.4.3 Time evolution of the function φ in the tilde domain

Recall that we have defined an auxiliary function $\varphi(\alpha,t)$ adapted to the tilde domain, which helps us to bound several of the terms that appear:

$$\varphi(\alpha,t) = \frac{Q^2(\alpha,t)\omega(\alpha,t)}{2|z_\alpha(\alpha,t)|} - c(\alpha,t)|z_\alpha(\alpha,t)|. \tag{2.62}$$

We will show how to find the evolution equation for φ_t. We have

$$\varphi = \frac{Q^2\omega}{2|z_\alpha|} - c|z_\alpha|$$

and therefore

$$\frac{\varphi^2}{Q^2} = \frac{Q^2\omega^2}{4|z_\alpha|^2} + \frac{c^2|z_\alpha|^2}{Q^2} - c\omega$$

that yields

$$-\partial_\alpha\left(\frac{\varphi^2}{Q^2}\right) = -\partial_\alpha\left(\frac{Q^2\omega^2}{4|z_\alpha|^2}\right) - \partial_\alpha\left(\frac{c^2|z_\alpha|^2}{Q^2}\right) + \partial_\alpha(c\omega).$$

The equation for ω_t reads:

$$\omega_t = -2BR_t\cdot z_\alpha - 2QQ_\alpha|BR|^2 + \underbrace{2cBR_\alpha\cdot z_\alpha}_{(1a)} - \partial_\alpha\left(\frac{\varphi^2}{Q^2}\right)\underbrace{+\partial_\alpha\left(\frac{c^2|z_\alpha|^2}{Q^2}\right)}_{(1b)} - 2\partial_\alpha\left(gP_2^{-1}\right). \quad (2.63)$$

For the quantity $(1) = (1a) + (1b)$ we write

$$\begin{aligned}(1) = (1a) + (1b) &= 2cBR_\alpha\cdot z_\alpha + \partial_\alpha\left(\frac{c^2|z_\alpha|^2}{Q^2}\right) = 2c(BR_\alpha\cdot z_\alpha + \frac{c_\alpha|z_\alpha|^2}{Q^2} - \frac{c|z_\alpha|^2Q_\alpha}{Q^3})\\ &= 2c[BR_\alpha\cdot z_\alpha + \frac{|z_\alpha|^2}{2\pi Q^2}\int_{-\pi}^{\pi}(Q^2BR)_\beta\cdot\frac{z_\beta}{|z_\beta|^2}d\beta - \frac{(Q^2BR)_\alpha\cdot z_\alpha}{Q^2} - \frac{c|z_\alpha|^2Q_\alpha}{Q^3}]\\ &= 2c[\frac{|z_\alpha|^2}{2\pi Q^2}\int_{-\pi}^{\pi}(Q^2BR)_\beta\cdot\frac{z_\beta}{|z_\beta|^2}d\beta - \frac{2Q_\alpha BR\cdot z_\alpha}{Q} - \frac{c|z_\alpha|^2Q_\alpha}{Q^3}]\end{aligned}$$

and then (2.63) becomes

$$\begin{aligned}\omega_t =& -2BR_t\cdot z_\alpha - 2QQ_\alpha|BR|^2 - \partial_\alpha\left(\frac{\varphi^2}{Q^2}\right)\\ &+ \frac{c|z_\alpha|^2}{\pi Q^2}\int_{-\pi}^{\pi}(Q^2BR)_\beta\cdot\frac{z_\beta}{|z_\beta|^2}d\beta - \frac{4cQ_\alpha BR\cdot z_\alpha}{Q} - \frac{2c^2|z_\alpha|^2Q_\alpha}{Q^3} - 2\partial_\alpha\left(gP_2^{-1}\right).\end{aligned} \quad (2.64)$$

Furthermore

$$\begin{aligned}\varphi_t &= QQ_t\frac{\omega}{|z_\alpha|} - \frac{Q^2\omega}{2|z_\alpha|^3}z_\alpha\cdot z_{\alpha t} + \frac{Q^2\omega_t}{2|z_\alpha|} - \partial_t(c|z_\alpha|)\\ &= QQ_t\frac{\omega}{|z_\alpha|} - \frac{Q^2\omega}{2|z_\alpha|}\frac{1}{2\pi}\int_{-\pi}^{\pi}(Q^2BR)_\beta\cdot\frac{z_\beta}{|z_\beta|^2}d\beta\\ &\quad + \frac{Q^2}{2|z_\alpha|}\Bigg[-2BR_t\cdot z_\alpha - 2QQ_\alpha|BR|^2 - \partial_\alpha\left(\frac{\varphi^2}{Q^2}\right)\\ &\quad + \frac{c|z_\alpha|^2}{\pi Q^2}\int_{-\pi}^{\pi}(Q^2BR)_\beta\cdot\frac{z_\beta}{|z_\beta|^2}d\beta - \frac{4cQ_\alpha BR\cdot z_\alpha}{Q} - \frac{2c^2|z_\alpha|^2Q_\alpha}{Q^3} - 2\partial_\alpha\left(gP_2^{-1}\right)\Bigg] - \partial_t(c|z_\alpha|).\end{aligned}$$

We should remark that we have used that

$$z_\alpha \cdot z_{\alpha t} = \frac{1}{2\pi}\int_{-\pi}^{\pi}(Q^2BR)_\beta \cdot z_\beta d\beta.$$

For simplicity, we denote

$$B(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(Q^2BR)_\beta \cdot \frac{z_\beta}{|z_\beta|^2}d\beta. \tag{2.65}$$

Computing

$$\begin{aligned}\varphi_t = QQ_t\frac{\omega}{|z_\alpha|}\underbrace{-\frac{Q^2\omega}{2|z_\alpha|}B(t)}_{(2a)}-\frac{Q^2}{|z_\alpha|}BR_t\cdot z_\alpha-\frac{Q^3Q_\alpha}{|z_\alpha|}|BR|^2-\frac{Q^2}{2|z_\alpha|}\partial_\alpha\left(\frac{\varphi^2}{Q^2}\right)\\ \underbrace{+c|z_\alpha|B(t)}_{(2b)}-\frac{2cQQ_\alpha}{|z_\alpha|}BR\cdot z_\alpha-\frac{c^2|z_\alpha|Q_\alpha}{Q}-\frac{Q^2}{|z_\alpha|}\partial_\alpha\left(gP_2^{-1}\right)-\partial_t(c|z_\alpha|)\end{aligned}$$

We can write

$$(2) = (2a) + (2b) = -B(t)\varphi,$$

and it yields

$$\begin{aligned}\varphi_t = -\varphi B(t) - \frac{Q^2}{2|z_\alpha|}\partial_\alpha\left(\frac{\varphi^2}{Q^2}\right) - Q^2\Big(BR_t\cdot\frac{z_\alpha}{|z_\alpha|}+\frac{\partial_\alpha\left(gP_2^{-1}\right)}{|z_\alpha|}\Big)\\ + QQ_t\frac{\omega}{|z_\alpha|} - 2cBR\cdot\frac{z_\alpha}{|z_\alpha|}QQ_\alpha - \frac{Q_\alpha}{Q}c^2|z_\alpha| - \frac{Q^3}{|z_\alpha|}Q_\alpha|BR|^2 - \partial_t(c|z_\alpha|).\end{aligned} \tag{2.66}$$

We will use the equation above to perform energy estimates.

2.4.4 Definition and a priori estimates of the energy in the tilde domain

Let us consider for $k \geq 4$ the following definition of energy $E(t)$:

$$\begin{aligned}E(t) = 1 + \|z\|^2_{H^{k-1}}(t) + \int_{-\pi}^{\pi}\frac{Q^2(z)\sigma}{|z_\alpha|^2}|\partial_\alpha^k z|^2 d\alpha + \|\mathcal{F}(z)\|^2_{L^\infty}(t)\\ + \|\omega\|^2_{H^{k-2}}(t) + \|\varphi\|^2_{H^{k-\frac{1}{2}}}(t) + \frac{|z_\alpha|^2}{m(Q^2\sigma)(t)} + \sum_{l=0}^{4}\frac{1}{m(q^l)(t)},\end{aligned} \tag{2.67}$$

where

$$\mathcal{F}(z) = \frac{|\beta|}{|z(\alpha)-z(\alpha-\beta)|}, \quad \alpha,\beta\in[-\pi,\pi],$$

and $m(Q^2\sigma) = \min_{\alpha\in\mathbb{T}}\{Q^2(z(\alpha,t))\sigma(\alpha,t)\}$. In the next section we shall show a proof of the following lemma.

Lemma 2.4.1 *Let $z(\alpha,t)$ and $\omega(\alpha,t)$ be a solution of (3.1-3.2). Then, the following a priori estimate holds:*

$$\frac{d}{dt}E(t) \leq CE^p(t), \tag{2.68}$$

for $k \geq 4$ and C and p constants depending only on k.

The following subsections are devoted to proving Lemma 2.4.1 by showing the regularity of the different elements involved in the problem: the Birkhoff-Rott integral, $z_t(\alpha,t)$, $\omega_t(\alpha,t)$, $\omega(\alpha,t)$; $BR_t(\alpha,t)$, the R-T function $\sigma(\alpha,t)$ and its time derivative $\sigma_t(\alpha,t)$.

2.4.4.1 Estimates for BR

In this section we show that the Birkhoff-Rott integral is as regular as $\partial_\alpha z$.

Lemma 2.4.2 *The following estimate holds*

$$\|BR(z,\omega)\|_{H^k} \leq C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+1}} + \|\omega\|^2_{H^k})^j, \tag{2.69}$$

for $k \geq 2$, where C and j are constants independent of z and ω.

Remark 2.4.3 *Using this estimate for $k = 2$ we find easily that*

$$\|\partial_\alpha BR(z,\omega)\|_{L^\infty} \leq C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^3} + \|\omega\|^2_{H^2})^j, \tag{2.70}$$

which shall be used throughout the paper, where C and j are universal constants.

Proof: The proof can be done as in [28, Section 6.1] since the definition for the Birkhoff-Rott operator is independent of the domain. □

2.4.4.2 Estimates for z_t

In this section we show that z_t is as regular as $\partial_\alpha z$.

Lemma 2.4.4 *The following estimate holds*

$$\|z_t\|_{H^k} \leq C\left(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+1}} + \|\omega\|^2_{H^k} + \sum_{l=0}^{4}\frac{1}{m(q^l)(t)}\right)^j, \tag{2.71}$$

for $k \geq 2$, where C and j are constants that depend only on k.

Proof: It follows from [28, Section 6.2]. The only additional thing we need to control is an L^∞ norm of Q^2, which we can easily bound by the $m(q^l)$ terms which control the distance from the curve to the q^l points, more precisely, the one that controls the distance from the origin. □

2.4.4.3 Estimates for ω_t

This section is devoted to showing that ω_t is as regular as $\partial_\alpha\omega$.

Lemma 2.4.5 *The following estimate holds*

$$\|\omega_t\|_{H^k} \leq C\left(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+2}} + \|\omega\|^2_{H^{k+1}} + \|\varphi\|^2_{H^{k+1}} + \sum_{l=0}^{4}\frac{1}{m(q^l)(t)}\right)^j, \tag{2.72}$$

for $k \geq 1$, where C and j are constants that depend only on k.

Proof: We use formula (2.64) and proceed as in [28, Section 6.3]. Note that in [28] an exponential growth appears in the bound of the estimates for the nonlocal operator acting on ω_t (see equation (2.64)). However, in a recent paper [30] the authors get a polynomial growth for the operator in both 2 and 3 dimensions. Note that even the exponential growth is still good enough to prove Theorem 3.5.1.

□

2.4.4.4 Estimates for ω

In this section we show that the amplitude of the vorticity ω lies at the same level as $\partial_\alpha z$. We shall consider $z \in H^k(\mathbb{T})$, $\varphi \in H^{k-\frac{1}{2}}(\mathbb{T})$ and $\omega \in H^{k-2}(\mathbb{T})$ as part of the energy estimates. The inequality below yields $\omega \in H^{k-1}(\mathbb{T})$.

Lemma 2.4.6 *The following estimate holds*

$$\|\omega\|_{H^{k-1}} \leq C\left(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k}} + \|\omega\|^2_{H^{k-2}} + \|\varphi\|^2_{H^{k-1}} + \sum_{l=0}^{4}\frac{1}{m(q^l)(t)}\right)^j, \tag{2.73}$$

for $k \geq 3$, where C and j are constants that depend only on k.

Proof: We can apply the same techniques as in [28, Section 6.4] since the most singular terms are treated there and the other terms are harmless and can be easily estimated. The impact of Q is now taken into account by the $m(q^l)$ terms (which now cover all of the points $q^0, ..., q^4$). □

2.4.4.5 Estimates for BR_t.

Here we prove that the time derivative of the Birkhoff-Rott integral is at the same level as $\partial^2_\alpha z$.

Lemma 2.4.7 *The following estimate holds*

$$\|BR_t\|_{H^k} \leq C\left(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+2}} + \|\omega\|^2_{H^{k+1}} + \|\varphi\|^2_{H^{k+1}} + \sum_{l=0}^{4}\frac{1}{m(q^l)(t)}\right)^j, \tag{2.74}$$

for $k \geq 2$, where C and j are constants that depend only on k.

Proof: We proceed as in [28, Section 6.5], where BR_t appears in the formula (5.2). We use (2.71) and (2.72) to bound z_t and ω_t in BR_t respectively. □

2.4.4.6 Estimates for the Rayleigh-Taylor function σ

Here we prove that the Rayleigh-Taylor function is at the same level as $\partial_\alpha^2 z$.

Lemma 2.4.8 *The following estimate holds*

$$\|\sigma\|_{H^k} \leq C\left(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+2}} + \|\omega\|^2_{H^{k+1}} + \|\varphi\|^2_{H^{k+1}} + \sum_{l=0}^{4}\frac{1}{m(q^l)(t)}\right)^j, \tag{2.75}$$

for $k \geq 2$, where C and j are constants that depend only on k.

Proof: We proceed as in [28, Section 6.5] using formula (5.2). There is a new term in the definition of σ, namely $Q\left|BR(z,\omega) + \frac{\omega}{|z_\alpha|^2}z_\alpha\right|^2(\nabla Q)(z)\cdot z_\alpha^\perp$, but this term is less singular than $BR_t(z,\omega)\cdot z_\alpha^\perp$. Hence, the new term causes no trouble. □

2.4.4.7 Estimates for σ_t

In this section we obtain an upper bound for the L^∞ norm of σ_t that will be used in the energy inequalities and in the treatment of the Rayleigh-Taylor condition.

Lemma 2.4.9 *The following estimate holds*

$$\|\sigma_t\|_{L^\infty} \leq C\left(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^4} + \|\omega\|^2_{H^3} + \|\varphi\|^2_{H^3} + \sum_{l=0}^{4}\frac{1}{m(q^l)(t)}\right)^j, \tag{2.76}$$

where C and j are universal constants.

Proof: Again, as in the previous subsection, the new term is less singular than the terms treated in [28, Section 6.6]. Hence we deal with them with no problem. □

2.4.4.8 Energy estimates on the curve

In this section we give the proof of the following lemma when, again, $k = 4$. The case $k > 4$ is left to the reader. Regarding $\|\partial_\alpha^4 z\|^2_{L^2}$ let us remark that we have

$$\|\partial_\alpha^4 z\|^2_{L^2}(t) = \int_{\mathbb{T}}\frac{Q^2\sigma|z_\alpha|^2}{Q^2\sigma|z_\alpha|^2}|\partial_\alpha^4 z|^2 d\alpha \leq \frac{|z_\alpha|^2}{m(Q^2\sigma)(t)}\int_{\mathbb{T}}\frac{Q^2\sigma}{|z_\alpha|^2}|\partial_\alpha^4 z|^2 d\alpha. \tag{2.77}$$

Lemma 2.4.10 *Let $z(\alpha,t)$ and $\omega(\alpha,t)$ be a solution of (3.1-3.2). Then, the following a priori estimate holds:*

$$\frac{d}{dt}\left(\|z\|^2_{H^{k-1}} + \int_{-\pi}^{\pi}\frac{Q^2\sigma}{|z_\alpha|^2}|\partial_\alpha^k z|^2 d\alpha + \|\mathcal{F}(z)\|^2_{L^\infty}\right) \leq S(t) + CE^p(t), \tag{2.78}$$

for

$$S(t) = \int_{-\pi}^{\pi} 2Q^2\sigma\frac{\partial_\alpha^k z\cdot z_\alpha^\perp}{|z_\alpha|^3}\Lambda(\partial_\alpha^{k-1}\varphi)d\alpha, \tag{2.79}$$

and $k \geq 4$, where C and p are constants that depend only on k.

(The term $S(t)$ is uncontrolled but it will appear in the equation of the evolution of φ with the opposite sign.)

Proof: Using (2.71) and (2.77) one gets easily

$$\begin{aligned}\frac{d}{dt}\|z\|^2_{H^3} \leq & C\int_{-\pi}^{\pi}(|z(\alpha)||z_t(\alpha)| + |\partial^3_\alpha z(\alpha)||\partial^3_\alpha z_t(\alpha)|)d\alpha\\ \leq & CE^p(t).\end{aligned}$$

We obtain

$$\frac{d}{dt}\|\mathcal{F}(z)\|^2_{L^\infty} \leq CE^p(t)$$

in a similar manner as in [28, Section 7.2]. It remains to deal with the quantity

$$\begin{aligned}\frac{d}{dt}\int_{-\pi}^{\pi}\frac{Q^2\sigma}{|z_\alpha|^2}|\partial^4_\alpha z|^2 d\alpha =& \int_{-\pi}^{\pi}\Big(\frac{Q^2\sigma}{|z_\alpha|^2}\Big)_t|\partial^4_\alpha z|^2 d\alpha + \int_{-\pi}^{\pi}\frac{2Q^2\sigma}{|z_\alpha|^2}\partial^4_\alpha z\cdot\partial^4_\alpha z_t d\alpha\\ =& I_1 + I_2.\end{aligned}$$

The bounds (2.71), (2.77) and (2.76) give us

$$I_1 \leq CE^p(t).$$

Next for I_2 we write

$$\begin{aligned}I_2 =& \int_{-\pi}^{\pi}\frac{2Q^2\sigma}{|z_\alpha|^2}\partial^4_\alpha z\cdot\partial^4_\alpha(Q^2BR)d\alpha + \int_{-\pi}^{\pi}\frac{2Q^2\sigma}{|z_\alpha|^2}\partial^4_\alpha z\cdot\partial^4_\alpha(c\partial_\alpha z)d\alpha\\ =& J_1 + J_2.\end{aligned}$$

The most singular terms in J_1 are given by K_1, K_2, K_3 and K_4:

$$K_1 = \frac{1}{\pi}PV\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{Q^4\sigma}{|z_\alpha|^2}\partial^4_\alpha z\cdot\frac{(\partial^4_\alpha z - \partial^4_\alpha z')^\perp}{|z-z'|^2}\omega' d\beta d\alpha,$$

$$K_2 = -\frac{2}{\pi}PV\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{Q^4\sigma}{|z_\alpha|^2}\partial^4_\alpha z\cdot\frac{(z-z')^\perp}{|z-z'|^4}(z-z')\cdot(\partial^4_\alpha z - \partial^4_\alpha z')\omega' d\beta d\alpha,$$

$$K_3 = \frac{1}{\pi}PV\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{Q^4\sigma}{|z_\alpha|^2}\partial^4_\alpha z\cdot\frac{(z-z')^\perp}{|z-z'|^2}\partial^4_\alpha\omega' d\beta d\alpha,$$

and

$$K_4 = \frac{2}{\pi}\int_{-\pi}^{\pi}\frac{Q^3\sigma}{|z_\alpha|^2}\partial^4_\alpha z\cdot BR\,\nabla Q(z)\cdot\partial^4_\alpha z d\alpha,$$

where the prime denotes a function in the variable $\alpha-\beta$, i.e. $f' = f(\alpha-\beta)$.

Then we write:

$$\begin{aligned}
K_1 &= \frac{1}{\pi} PV \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{Q^4(\alpha)\sigma(\alpha)}{|z_\alpha|^2}\partial_\alpha^4 z(\alpha)\cdot\frac{(\partial_\alpha^4 z(\alpha)-\partial_\alpha^4 z(\beta))^\perp}{|z(\alpha)-z(\beta)|^2}\omega(\beta)d\beta d\alpha\\
&= \frac{1}{\pi|z_\alpha|^2} PV \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \partial_\alpha^4 z(\alpha)\cdot\frac{(\partial_\alpha^4 z(\alpha)-\partial_\alpha^4 z(\beta))^\perp}{|z(\alpha)-z(\beta)|^2}\frac{Q^4(\alpha)\sigma(\alpha)\omega(\beta)+Q^4(\beta)\sigma(\beta)\omega(\alpha)}{2}d\beta d\alpha\\
&\quad+ \frac{1}{\pi|z_\alpha|^2} PV \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \partial_\alpha^4 z(\alpha)\cdot\frac{(\partial_\alpha^4 z(\alpha)-\partial_\alpha^4 z(\beta))^\perp}{|z(\alpha)-z(\beta)|^2}\frac{Q^4(\alpha)\sigma(\alpha)\omega(\beta)-Q^4(\beta)\sigma(\beta)\omega(\alpha)}{2}d\beta d\alpha\\
&= L_1 + L_2.
\end{aligned}$$

That is, we have performed a manipulation in K_1, allowing us to show that L_1, its most singular term, vanishes:

$$\begin{aligned}
L_1 &= \frac{-1}{\pi|z_\alpha|^2} PV \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \partial_\alpha^4 z(\beta)\cdot\frac{(\partial_\alpha^4 z(\alpha)-\partial_\alpha^4 z(\beta))^\perp}{|z(\alpha)-z(\beta)|^2}\frac{Q^4(\alpha)\sigma(\alpha)\omega(\beta)+Q^4(\beta)\sigma(\beta)\omega(\alpha)}{2}d\beta d\alpha\\
&= \frac{1}{2\pi|z_\alpha|^2} PV \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} (\partial_\alpha^4 z(\alpha)-\partial_\alpha^4 z(\beta))\cdot\frac{(\partial_\alpha^4 z(\alpha)-\partial_\alpha^4 z(\beta))^\perp}{|z(\alpha)-z(\beta)|^2}\frac{Q^4(\alpha)\sigma(\alpha)\omega(\beta)+Q^4(\beta)\sigma(\beta)\omega(\alpha)}{2}d\beta d\alpha\\
&= 0.
\end{aligned}$$

The term L_2 involves a S.I.O. (Singular Integral Operator) acting on $\partial_\alpha^4 z(\alpha)$ thanks to the minus sign between the two terms $Q^4\sigma\omega$. One can show that

$$L_2 \le C\|\mathcal{F}(z)\|_{L^\infty}^2\|z\|_{H^3}^k\|\omega\|_{C^{1,\delta}}\|\sigma\|_{C^{1,\delta}}\|Q^4\|_{C^{1,\delta}}\|\partial_\alpha^4 z\|_{L^2}^2 \le CE^p(t).$$

Inside K_2 we find that $(z-z')\cdot(\partial_\alpha^4 z-\partial_\alpha^4 z')$ can be written as follows:

$$\begin{aligned}
(z-z')\cdot(\partial_\alpha^4 z-\partial_\alpha^4 z') &= (z-z'-z_\alpha\beta)\cdot(\partial_\alpha^4 z-\partial_\alpha^4 z')\\
&\quad-\beta(z_\alpha-z_\alpha')\cdot\partial_\alpha^4 z'\\
&\quad+\beta(z_\alpha\cdot\partial_\alpha^4 z-z_\alpha'\cdot\partial_\alpha^4 z'),
\end{aligned}\tag{2.80}$$

then using that

$$z_\alpha\cdot\partial_\alpha^4 z = -3\partial_\alpha^2 z\cdot\partial_\alpha^3 z,\tag{2.81}$$

we can split K_2 as a sum of S.I.O.s operating on $\partial_\alpha^4 z(\alpha)$, plus a kernel of the form $\frac{\eta(\alpha,\beta)}{\beta^2}$ acting on $\partial_\alpha^2 z\cdot\partial_\alpha^3 z$ with $\eta\in\mathcal{C}^2$ allowing us to obtain again the estimate

$$K_2 \le CE^p(t).$$

Note that below we will also use a variant of (2.81), namely

$$z_\alpha\cdot(\partial_\alpha^4 z-\partial_\alpha^4 z') = (z_\alpha'-z_\alpha)\cdot\partial_\alpha^4 z' - 3(\partial_\alpha^2 z\cdot\partial_\alpha^3 z-\partial_\alpha^2 z'\cdot\partial_\alpha^3 z').\tag{2.82}$$

The term K_3 is a sum of

$$L_3 = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{Q^4\sigma}{|z_\alpha|^2}\partial_\alpha^4 z\cdot\int_{-\pi}^{\pi}\left[\frac{(z-z')^\perp}{|z-z'|^2}-\frac{z_\alpha^\perp}{|z_\alpha|^2 2\tan(\beta/2)}\right]\partial_\alpha^4\omega' d\beta d\alpha,$$

plus the following term:

$$L_4 = \int_{-\pi}^{\pi} Q^4 \sigma \frac{\partial_\alpha^4 z \cdot z_\alpha^\perp}{|z_\alpha|^4} H(\partial_\alpha^4 \omega) d\alpha.$$

We can integrate by parts on L_3 with respect to β since $\partial_\alpha^4 \omega' = -\partial_\beta(\partial_\alpha^3 \omega')$. This calculation gives a S.I.O. acting on $\partial_\alpha^3 \omega$ which can be estimated as before.

Next in L_4 we write

$$L_4 = \int_{-\pi}^{\pi} Q^4 \sigma \frac{\partial_\alpha^4 z \cdot z_\alpha^\perp}{|z_\alpha|^4} \Lambda(\partial_\alpha^3 \omega) d\alpha$$

and decompose further

$$\begin{aligned} L_4 =& \int_{-\pi}^{\pi} 2Q^2 \sigma \frac{\partial_\alpha^4 z \cdot z_\alpha^\perp}{|z_\alpha|^3} \left[\frac{Q^2}{2|z_\alpha|} \Lambda(\partial_\alpha^3 \omega) - \Lambda(\partial_\alpha^3 (\frac{Q^2}{2|z_\alpha|} \omega)) \right] d\alpha \\ &+ \int_{-\pi}^{\pi} 2Q^2 \sigma \frac{\partial_\alpha^4 z \cdot z_\alpha^\perp}{|z_\alpha|^3} \Lambda(\partial_\alpha^3 \varphi) d\alpha \\ &+ \int_{-\pi}^{\pi} 2Q^2 \sigma \frac{\partial_\alpha^4 z \cdot z_\alpha^\perp}{|z_\alpha|^3} \Lambda(\partial_\alpha^3 (c|z_\alpha|)) d\alpha \\ =& M_{-1} + S + M_1, \end{aligned}$$

for $S(t)$ given by (2.79). In M_{-1} we find a commutator that allows us to obtain

$$M_{-1} \leq C E^p(t).$$

Using (2.61) for M_1 we have

$$M_1 = -2 \int_{-\pi}^{\pi} H\big(Q^2 \sigma \frac{\partial_\alpha^4 z \cdot z_\alpha^\perp}{|z_\alpha|^3}\big) \partial_\alpha^4 (c|z_\alpha|)) d\alpha = N_1 + N_2 + N_3 + N_4,$$

where

$$\begin{aligned} N_1 &= 2 \int_{-\pi}^{\pi} H\big(Q^2 \sigma \frac{\partial_\alpha^4 z \cdot z_\alpha^\perp}{|z_\alpha|^3}\big) Q^2 BR_\alpha \cdot \frac{\partial_\alpha^4 z}{|z_\alpha|} d\alpha, \\ N_2 &= 2 \int_{-\pi}^{\pi} H\big(Q^2 \sigma \frac{\partial_\alpha^4 z \cdot z_\alpha^\perp}{|z_\alpha|^3}\big) Q^2 \partial_\alpha^4 BR \cdot \frac{z_\alpha}{|z_\alpha|} d\alpha, \\ N_3 &= 4 \int_{-\pi}^{\pi} H\big(Q^2 \sigma \frac{\partial_\alpha^4 z \cdot z_\alpha^\perp}{|z_\alpha|^3}\big) Q \nabla Q \cdot \partial_\alpha^4 z BR \cdot \frac{z_\alpha}{|z_\alpha|} d\alpha, \end{aligned}$$

and N_4 is given by the rest of the terms which can be controlled easily by the estimates from Section 2.4.4.1 for the Birkhoff-Rott integral.

Regarding N_1 a straightforward calculation gives

$$N_1 \leq C E^p(t),$$

and analogously for N_3

$$N_3 \leq C E^p(t).$$

Again, in N_2 we consider the most singular terms given by

$$O_1 = \int_{-\pi}^{\pi} H\big(Q^2\sigma\frac{\partial_\alpha^4 z\cdot z_\alpha^\perp}{|z_\alpha|^3}\big)\frac{z_\alpha}{|z_\alpha|}\cdot\frac{Q^2}{\pi}PV\int_{-\pi}^{\pi}\frac{(\partial_\alpha^4 z-\partial_\alpha^4 z')^\perp}{|z-z'|^2}\omega' d\beta d\alpha,$$

$$O_2 = -\int_{-\pi}^{\pi} H\big(Q^2\sigma\frac{\partial_\alpha^4 z\cdot z_\alpha^\perp}{|z_\alpha|^3}\big)\frac{z_\alpha}{|z_\alpha|}\cdot\frac{Q^2}{2\pi}PV\int_{-\pi}^{\pi}\frac{(z-z')^\perp}{|z-z'|^4}(z-z')\cdot(\partial_\alpha^4 z-\partial_\alpha^4 z')\omega' d\alpha d\beta,$$

$$O_3 = 2\int_{-\pi}^{\pi} H\big(Q^2\sigma\frac{\partial_\alpha^4 z\cdot z_\alpha^\perp}{|z_\alpha|^3}\big)Q^2\frac{z_\alpha}{|z_\alpha|}\cdot BR(z,\partial_\alpha^4\omega)d\alpha.$$

Using the decomposition (2.80) we can easily estimate O_2 as in our discussion of K_2.

In O_3 we find

$$z_\alpha\cdot BR(z,\partial_\alpha^4\omega) = \frac{z_\alpha}{2\pi}\cdot\int_{-\pi}^{\pi}\frac{(z-z'-z_\alpha\beta)^\perp}{|z-z'|^2}\partial_\alpha^4\omega' d\beta.$$

Above we can integrate by parts as in our discussion of L_3. We find that

$$O_3 \le CE^p(t).$$

Next we split O_1 into a S.I.O. acting on $(\partial_\alpha^4 z)^\perp$, which can be estimated as before, plus the term

$$P_1 = \int_{-\pi}^{\pi} H\big(Q^2\sigma\frac{\partial_\alpha^4 z\cdot z_\alpha^\perp}{|z_\alpha|^3}\big)Q^2\omega\frac{z_\alpha}{|z_\alpha|^3}\cdot\Lambda((\partial_\alpha^4 z)^\perp)d\alpha.$$

Then the following estimate for the commutator

$$\|Q^2\omega\frac{z_\alpha}{|z_\alpha|^3}\cdot\Lambda((\partial_\alpha^4 z)^\perp)-\Lambda(Q^2\omega\frac{z_\alpha}{|z_\alpha|^3}\cdot(\partial_\alpha^4 z)^\perp)\|_{L^2}\le CE^p(t)$$

yields

$$P_1 \le CE^p(t)+R$$

where

$$R = -\int_{-\pi}^{\pi} Q^2\sigma\frac{\partial_\alpha^4 z\cdot z_\alpha^\perp}{|z_\alpha|^3}\partial_\alpha(Q^2\omega\frac{z_\alpha}{|z_\alpha|^3}(\partial_\alpha^4 z)^\perp)d\alpha.$$

Using that

$$\int_{-\pi}^{\pi} Hf(\alpha)\Lambda g(\alpha)d\alpha = -\int_{-\pi}^{\pi} f(\alpha)\partial_\alpha g(\alpha)d\alpha.$$

We can write

$$R = \int_{-\pi}^{\pi} Q^2\sigma\frac{\partial_\alpha^4 z\cdot z_\alpha^\perp}{|z_\alpha|^3}\partial_\alpha(Q^2\omega)\frac{\partial_\alpha^4 z\cdot z_\alpha^\perp}{|z_\alpha|^3}d\alpha + \int_{-\pi}^{\pi} Q^2\sigma\frac{\partial_\alpha^4 z\cdot z_\alpha^\perp}{|z_\alpha|^3}Q^2\omega\partial_\alpha(\frac{\partial_\alpha^4 z\cdot z_\alpha^\perp}{|z_\alpha|^3})d\alpha$$

and a straightforward integration by parts let us control R. This calculation allows us to get

$$P_1 \le CE^p(t).$$

We can easily show that

$$K_4 \leq CE^p(t)$$

because we can bound $Q^3\sigma BR\nabla Q$ in L^∞. So finally we have controlled J_1 in the following manner:

$$J_1 \leq CE^p(t) + S.$$

To finish the proof let us observe that the term J_2 can be estimated integrating by parts, using the identity $\partial_\alpha^4 z \cdot \partial_\alpha z = -3\partial_\alpha^3 z \cdot \partial_\alpha^2 z$ to treat its most singular component. We have obtained

$$\int_{\mathbb{T}} \frac{Q^2\sigma}{|z_\alpha|^2}\partial_\alpha^4 z \cdot \partial_\alpha z \partial_\alpha^4 c d\alpha = 3\int_{\mathbb{T}} \frac{1}{|z_\alpha|^2}\partial_\alpha(Q^2\sigma\partial_\alpha^3 z \cdot \partial_\alpha^2 z)\partial_\alpha^3 c d\alpha$$

and this yields the desired control. □

2.4.4.9 Energy estimates for ω

In this section we show the following result.

Lemma 2.4.11 *Let $z(\alpha,t)$ and $\omega(\alpha,t)$ be a solution of (3.1-3.2). Then, the following a priori estimate holds:*

$$\frac{d}{dt}\|\omega\|^2_{H^{k-2}}(t) \leq CE^p(t), \tag{2.83}$$

for $k \geq 4$, where C and p are constants that depend only on k.

Proof: We will discuss the case $k = 4$, leaving the other cases to the reader. Formula (2.72) shows easily that

$$\frac{d}{dt}\|\omega\|^2_{H^2}(t) \leq \left(\|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|z\|^2_{H^4}(t) + \|\omega\|^2_{H^3}(t) + \|\varphi\|^2_{H^3}(t) + \sum_{l=0}^{4}\frac{1}{m(q^l)(t)}\right)^j$$

which together with (2.73) yields

$$\frac{d}{dt}\|\omega\|^2_{H^2}(t) \leq CE^p(t).$$

□

2.4.4.10 Finding the Rayleigh-Taylor function in the equation for $\partial_\alpha\varphi_t$.

In this section we get the R-T function in the evolution equation for $\partial_\alpha\varphi_t$.

Lemma 2.4.12 *Let $z(\alpha,t)$ and $\omega(\alpha,t)$ be a solution of (3.1-3.2). Then, the following identity holds:*

$$\varphi_{\alpha t} = NICE - \frac{\varphi}{|z_\alpha|}\varphi_{\alpha\alpha} - Q^2\sigma\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3} \tag{2.84}$$

where NICE satisfies

$$\int_{-\pi}^{\pi}\Lambda(\partial_\alpha^{k-1}\varphi)\partial_\alpha^{k-2}(NICE)d\alpha \leq CE^p(t) \tag{2.85}$$

and

$$\|NICE\|_{H^{k-2}} \leq CE^p(t) \tag{2.86}$$

for $k\geq 4$, where C and p are constants that depend only on k.

Proof: We will give the proof for $k=4$. From now on, when we show that a term f satisfies

$$\int_{-\pi}^{\pi}\Lambda(\partial_\alpha^3\varphi)\partial_\alpha^2 f d\alpha \leq CE^p(t) \quad \text{and } \|f\|_{H^2}\leq CE^p(t)$$

we say that this term is "NICE". Then, f becomes part of NICE and by abuse of notation we denote f by NICE. Notice that, whenever we can estimate the L^2 norm of $\Lambda^{1/2}\partial_\alpha^2 f$ by $CE^p(t)$, then f is NICE.

We use (2.66) to compute

$$\begin{aligned}\varphi_{\alpha t} =& -B(t)\varphi_\alpha - \partial_\alpha\left(\frac{Q^2}{2|z_\alpha|}\left(\frac{\varphi^2}{Q^2}\right)_\alpha\right) \underbrace{-\big(Q^2\big(BR_t\cdot\frac{z_\alpha}{|z_\alpha|}}_{(3a)} + \frac{(gP_2^{-1}(z))_\alpha}{|z_\alpha|}\big)\big)_\alpha \\ &+\left(QQ_t\frac{\omega}{|z_\alpha|}\right)_\alpha - \left(2cBR\cdot\frac{z_\alpha}{|z_\alpha|}QQ_\alpha\right)_\alpha - \left(\frac{Q_\alpha}{Q}c^2|z_\alpha|\right)_\alpha - \left(\frac{Q^3}{|z_\alpha|}|BR|^2Q_\alpha\right)_\alpha \\ &\underbrace{-(c|z_\alpha|)_{\alpha t}}_{(3b)}.\end{aligned}$$

Expanding $(3)=(3a)+(3b)$:

$$\begin{aligned}(3) =&(3a)+(3b) = -\left(Q^2BR_t\cdot\frac{z_\alpha}{|z_\alpha|}\right)_\alpha - (c|z_\alpha|)_{\alpha t} \\ =& -\big(Q^2BR_t\big)_\alpha\cdot\frac{z_\alpha}{|z_\alpha|} - Q^2BR_t\cdot\left(\frac{z_\alpha}{|z_\alpha|}\right)_\alpha - \left(|z_\alpha|B(t)-(Q^2BR)_\alpha\cdot\frac{z_\alpha}{|z_\alpha|}\right)_t \\ =& -Q^2BR_t\cdot\left(\frac{z_\alpha}{|z_\alpha|}\right)_\alpha - (|z_\alpha|B(t))_t + (Q^2BR)_\alpha\cdot\left(\frac{z_\alpha}{|z_\alpha|}\right)_t + 2(QQ_tBR)_\alpha\cdot\frac{z_\alpha}{|z_\alpha|}.\end{aligned}$$

We use that

$$\left(\frac{z_\alpha}{|z_\alpha|}\right)_\alpha = \frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^2}\cdot\frac{z_\alpha^\perp}{|z_\alpha|};\quad \left(\frac{z_\alpha}{|z_\alpha|}\right)_t = \frac{z_{\alpha t}\cdot z_\alpha^\perp}{|z_\alpha|^2}\cdot\frac{z_\alpha^\perp}{|z_\alpha|}$$

to find

$$\begin{aligned}\varphi_{\alpha t} = &\underbrace{-B(t)\varphi_\alpha}_{(4)} - \underbrace{\frac{\partial^2_\alpha(\varphi^2)}{2|z_\alpha|}}_{(5)} + \underbrace{\partial_\alpha\left(\frac{Q_\alpha}{|z_\alpha|Q}\varphi^2\right)}_{(6)} - Q^2BR_t\cdot z_\alpha^\perp\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3} - (|z_\alpha|B(t))_t\\
&+\underbrace{(Q^2BR)_\alpha\cdot z_\alpha^\perp\frac{z_{\alpha t}\cdot z_\alpha^\perp}{|z_\alpha|^3}}_{(13)} + \underbrace{2(QQ_tBR)_\alpha\cdot\frac{z_\alpha}{|z_\alpha|}}_{(7)} - \underbrace{\left(Q^2\frac{(gP_2^{-1}(z))_\alpha}{|z_\alpha|}\right)_\alpha}_{(8)} + \underbrace{\left(QQ_t\frac{\omega}{|z_\alpha|}\right)_\alpha}_{(9)}\\
&-\underbrace{\left(2cBR\cdot\frac{z_\alpha}{|z_\alpha|}QQ_\alpha\right)_\alpha}_{(10)} - \underbrace{\left(\frac{Q_\alpha}{Q}c^2|z_\alpha|\right)_\alpha}_{(11)} - \underbrace{\left(\frac{Q^3}{|z_\alpha|}|BR|^2Q_\alpha\right)_\alpha}_{(12)}.\end{aligned} \tag{2.87}$$

The term $(|z_\alpha|B(t))_t$ depends only on t so it is going to be part of NICE.

$$(4) = -B(t)\varphi_\alpha \text{ is NICE (at the level of } \varphi_\alpha).$$

$$(5) = -\frac{\partial^2_\alpha(\varphi^2)}{2|z_\alpha|} = -\frac{\varphi_\alpha^2}{|z_\alpha|} - \frac{\varphi}{|z_\alpha|}\varphi_{\alpha\alpha}.$$

The first term is at the level of φ_α so it is NICE. The second one is the transport term which appears in (2.98).

$$(6) = \partial_\alpha\left(\frac{Q_\alpha}{|z_\alpha|Q}\varphi^2\right) = -\frac{Q_\alpha^2\varphi^2}{|z_\alpha|Q^2} + \frac{2Q_\alpha\varphi\varphi_\alpha}{|z_\alpha|Q} + \frac{\varphi^2}{Q}\left(\frac{Q_\alpha}{|z_\alpha|}\right)_\alpha.$$

Above we find the first term at the level of z_α so it is NICE. The second term is at the level of φ_α so it is NICE. We write the last one as

$$\frac{\varphi^2}{Q}\left(\frac{Q_\alpha}{|z_\alpha|}\right)_\alpha = \frac{\varphi^2}{Q}z_\alpha\cdot\left(\nabla^2Q(z)\cdot\frac{z_\alpha}{|z_\alpha|}\right) + \frac{\varphi^2}{Q}\nabla Q\cdot z_\alpha^\perp\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}.$$

The first term is at the level of z_α so it is NICE. For the second term we have used that

$$\left(\frac{z_\alpha}{|z_\alpha|}\right)_\alpha = \frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^2}\cdot\frac{z_\alpha^\perp}{|z_\alpha|}.$$

Finally:

$$(6) = \text{NICE } + \frac{\varphi^2}{Q}\nabla Q\cdot z_\alpha^\perp\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}.$$

$$(7) = 2(QQ_tBR)_\alpha\cdot\frac{z_\alpha}{|z_\alpha|} = 2Q_\alpha Q_tBR\cdot\frac{z_\alpha}{|z_\alpha|} + 2Q\left(\frac{Q_t}{|z_\alpha|}\right)_\alpha BR\cdot z_\alpha + 2QQ_tBR_\alpha\cdot\frac{z_\alpha}{|z_\alpha|}.$$

The first term is at the level of $z_\alpha, z_t, BR\sim z_\alpha$ so it is NICE. We use that

$$\frac{Q_{t\alpha}}{|z_\alpha|} = \frac{Q_{\alpha t}}{|z_\alpha|} = \frac{(\nabla Q(z)\cdot z_\alpha)_t}{|z_\alpha|} = \left(\nabla Q(z)\cdot\frac{z_\alpha}{|z_\alpha|}\right)_t - \nabla Q(z)\cdot z_\alpha\left(\frac{1}{|z_\alpha|}\right)_t.$$

Using equation (2.65)

$$\frac{z_\alpha\cdot z_{\alpha t}}{|z_\alpha|^2} = B(t)$$

and

$$\left(\frac{z_\alpha}{|z_\alpha|}\right)_t = \frac{z_{\alpha t}\cdot z_\alpha^\perp}{|z_\alpha|^2}\cdot\frac{z_\alpha^\perp}{|z_\alpha|}$$

we find that

$$\begin{aligned}\frac{Q_{t\alpha}}{|z_\alpha|} =& z_t\cdot\left(\nabla^2 Q(z)\cdot\frac{z_\alpha}{|z_\alpha|}\right) + \nabla Q(z)\cdot z_\alpha^\perp\frac{z_{\alpha t}\cdot z_\alpha^\perp}{|z_\alpha|^3}\\ &+ \nabla Q(z)\cdot\frac{z_\alpha}{|z_\alpha|}B(t).\end{aligned} \tag{2.88}$$

That yields

$$\begin{aligned}(7) = 2(QQ_tBR)_\alpha\frac{z_\alpha}{|z_\alpha|} =& \text{NICE } + \underbrace{2QBR\cdot z_\alpha\, z_t\cdot\left(\nabla^2 Q(z)\cdot\frac{z_\alpha}{|z_\alpha|}\right)}_{\text{NICE (at the level of } z_\alpha, z_t, BR)}\\ &+ 2QBR\cdot z_\alpha\nabla Q(z)\cdot z_\alpha^\perp\frac{z_{\alpha t}\cdot z_\alpha^\perp}{|z_\alpha|^3} + \underbrace{2QBR\cdot z_\alpha\nabla Q(z)\cdot\frac{z_\alpha}{|z_\alpha|}B(t)}_{\text{NICE (at the level of } z_\alpha, z_t, BR)}\\ &+ 2QQ_tBR_\alpha\cdot\frac{z_\alpha}{|z_\alpha|}.\end{aligned}$$

Finally:

$$(7) = 2(QQ_tBR)_\alpha\frac{z_\alpha}{|z_\alpha|} = \text{NICE } + 2QBR\cdot z_\alpha\nabla Q(z)\cdot z_\alpha^\perp\frac{z_{\alpha t}\cdot z_\alpha^\perp}{|z_\alpha|^3} + 2QQ_tBR_\alpha\cdot\frac{z_\alpha}{|z_\alpha|}.$$

$$\begin{aligned}(8) =& -\left(Q^2\frac{(gP_2^{-1}(z))_\alpha}{|z_\alpha|}\right)_\alpha = -\left(Q^2\nabla gP_2^{-1}(z)\cdot\frac{z_\alpha}{|z_\alpha|}\right)_\alpha\\ =& -\underbrace{2Q\nabla Q\cdot z_\alpha\nabla gP_2^{-1}(z)\cdot\frac{z_\alpha}{|z_\alpha|}}_{\text{NICE (at the level of } z_\alpha)} - \underbrace{Q^2 z_\alpha\cdot\left(\nabla^2 gP_2^{-1}(z)\cdot\frac{z_\alpha}{|z_\alpha|}\right)}_{\text{NICE (at the level of } z_\alpha)}\\ &- Q^2\nabla gP_2^{-1}(z)\cdot z_\alpha^\perp\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3},\end{aligned}$$

which means

$$(8) = -\left(Q^2\frac{(gP_2^{-1}(z))_\alpha}{|z_\alpha|}\right)_\alpha = \text{NICE } - Q^2\nabla gP_2^{-1}(z)\cdot z_\alpha^\perp\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}.$$

Next

$$(9) = \left(QQ_t\frac{\omega}{|z_\alpha|}\right)_\alpha = \underbrace{Q_\alpha Q_t\frac{\omega}{|z_\alpha|}}_{\text{NICE (at the level of } z_\alpha, z_t)} + Q\frac{Q_{\alpha t}}{|z_\alpha|}\omega + QQ_t\left(\frac{\omega}{|z_\alpha|}\right)_\alpha.$$

We use (5.12) to deal with $\frac{Q_{\alpha t}}{|z_\alpha|}$. We find that

$$(9) = \left(QQ_t\frac{\omega}{|z_\alpha|}\right)_\alpha = \text{NICE } + Q\omega\nabla Q(z)\cdot z_\alpha^\perp\frac{z_{\alpha t}\cdot z_\alpha^\perp}{|z_\alpha|^3} + QQ_t\left(\frac{\omega}{|z_\alpha|}\right)_\alpha.$$

For the next term

$$(10) = -\left(2cBR\cdot\frac{z_\alpha}{|z_\alpha|}QQ_\alpha\right)_\alpha = \underbrace{-2cBR\cdot\frac{z_\alpha}{|z_\alpha|}Q_\alpha^2}_{\text{NICE as before}} - \left(2cBR\cdot\frac{z_\alpha}{|z_\alpha|}\right)_\alpha QQ_\alpha$$
$$- 2cBR\cdot z_\alpha Q\nabla Q(z)\cdot z_\alpha^\perp\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}\underbrace{-2cBR\cdot\frac{z_\alpha}{|z_\alpha|}Qz_\alpha\cdot(\nabla^2Q(z))\cdot z_\alpha}_{\text{NICE as before}}.$$

Therefore

$$(10) = -\left(2cBR\cdot\frac{z_\alpha}{|z_\alpha|}QQ_\alpha\right)_\alpha = \text{NICE } - \left(2cBR\cdot\frac{z_\alpha}{|z_\alpha|}\right)_\alpha QQ_\alpha$$
$$- 2cBR\cdot z_\alpha Q\nabla Q\cdot z_\alpha^\perp\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}.$$

Next

$$(11) = -\left(\frac{Q_\alpha}{Q}c^2|z_\alpha|\right)_\alpha = -\left(c^2|z_\alpha|\right)_\alpha\frac{Q_\alpha}{Q} - \frac{c^2|z_\alpha|^2}{Q}\nabla Q(z)\cdot z_\alpha^\perp\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}$$
$$- \frac{z_\alpha\cdot(\nabla^2Q(z)\cdot z_\alpha)}{Q}c^2|z_\alpha| + \frac{Q_\alpha^2}{Q^2}c^2|z_\alpha|.$$

The fact that the last two terms are NICE, allows us to find that

$$(11) = -\left(\frac{Q_\alpha}{Q}c^2|z_\alpha|\right)_\alpha = \text{NICE } - \left(c^2|z_\alpha|\right)_\alpha\frac{Q_\alpha}{Q} - \frac{c^2|z_\alpha|^2}{Q}\nabla Q(z)\cdot z_\alpha^\perp\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}.$$

Finally:

$$
\begin{aligned}
(12) = -\left(\frac{Q^3}{|z_\alpha|}|BR|^2 Q_\alpha\right)_\alpha = & \underbrace{-\frac{3Q^2 Q_\alpha^2 |BR|^2}{|z_\alpha|}}_{\text{NICE}} - \frac{Q^3}{|z_\alpha|}(|BR|^2)_\alpha Q_\alpha \\
& \underbrace{-\frac{Q^3}{|z_\alpha|}|BR|^2 z_\alpha \cdot (\nabla^2 Q(z)\cdot z_\alpha)}_{\text{NICE}} - Q^3|BR|^2\nabla Q(z)\cdot z_\alpha^\perp \frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}
\end{aligned}
$$

which implies that

$$
(12) = -\left(\frac{Q^3}{|z_\alpha|}|BR|^2 Q_\alpha\right)_\alpha = \text{NICE } - \frac{Q^3}{|z_\alpha|}(|BR|^2)_\alpha Q_\alpha - Q^3|BR|^2\nabla Q(z)\cdot z_\alpha^\perp \frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}.
$$

We gather all the formulas from (4) to (12), keeping term (13) unchanged. They yield:

$$
\begin{aligned}
\varphi_{\alpha t} = & \text{NICE} - \frac{\varphi}{|z_\alpha|}\varphi_{\alpha\alpha} \\
& + \underbrace{\frac{\varphi^2}{Q}\nabla Q(z)\cdot z_\alpha^\perp \frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}}_{(16a)} \underbrace{- Q^2 BR_t \cdot z_\alpha^\perp \frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}}_{(15a)} \underbrace{- Q^2 \nabla g P_2^{-1}(z)\cdot z_\alpha^\perp \frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}}_{(15b)} \\
& + \underbrace{Q\omega\nabla Q(z)\cdot z_\alpha^\perp \frac{z_{\alpha t}\cdot z_\alpha^\perp}{|z_\alpha|^3}}_{(18a)} + \underbrace{QQ_t\left(\frac{\omega}{|z_\alpha|}\right)_\alpha}_{(14a)} + \underbrace{2QBR\cdot z_\alpha \nabla Q(z)\cdot z_\alpha^\perp \frac{z_{\alpha t}\cdot z_\alpha^\perp}{|z_\alpha|^3}}_{(18b)} \\
& + \underbrace{QQ_t 2BR_\alpha \cdot \frac{z_\alpha}{|z_\alpha|}}_{(14b)} \underbrace{- \left(2cBR\cdot \frac{z_\alpha}{|z_\alpha|}\right)_\alpha QQ_\alpha}_{(17a)} \underbrace{- 2cBR\cdot z_\alpha Q\nabla Q(z)\cdot z_\alpha^\perp \frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}}_{(16b)} \\
& \underbrace{-(c^2|z_\alpha|)_\alpha \frac{Q_\alpha}{Q}}_{(17b)} \underbrace{- \frac{c^2|z_\alpha|^2}{Q}\nabla Q(z)\cdot z_\alpha^\perp \frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}}_{(16c)} \underbrace{- \frac{Q^3}{|z_\alpha|}(|BR|^2)_\alpha Q_\alpha}_{(17c)} \\
& \underbrace{- Q^3|BR|^2\nabla Q(z)\cdot z_\alpha^\perp \frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}}_{(16d)} + (Q^2 BR)_\alpha \cdot z_\alpha^\perp \frac{z_{\alpha t}\cdot z_\alpha^\perp}{|z_\alpha|^3}.
\end{aligned}
$$

We compute

$$
\begin{aligned}
(14) = (14a) + (14b) &= QQ_t\left(\frac{\omega}{|z_\alpha|}\right)_\alpha + QQ_t 2BR_\alpha \cdot \frac{z_\alpha}{|z_\alpha|} \\
&= 2\frac{Q_t}{Q}Q^2\left(\frac{\omega}{2|z_\alpha|}\right)_\alpha + 2\frac{Q_t}{Q}Q^2 BR_\alpha \cdot \frac{z_\alpha}{|z_\alpha|} \\
&= 2\frac{Q_t}{Q}\varphi_\alpha - 2\frac{Q_t}{Q}(Q^2)_\alpha \frac{\omega}{2|z_\alpha|} - 2\frac{Q_t}{Q}(Q^2)_\alpha BR\cdot \frac{z_\alpha}{|z_\alpha|} + 2\frac{Q_t}{Q}(|z_\alpha|B(t)).
\end{aligned}
$$

The last formula allows us to conclude that (14)=NICE.

We reorganize gathering

$$(15) = (15a) + (15b),$$

$$(16) = (16a) + (16b) + (16c) + (16d),$$

$$(17) = (17a) + (17b) + (17c)$$

and

$$(18) = (18a) + (18b)$$

as follows:

$$\begin{aligned}
\varphi_{\alpha t} =& \text{NICE } - \frac{\varphi}{|z_\alpha|}\varphi_{\alpha\alpha} \underbrace{-Q^2(BR_t \cdot z_\alpha^\perp + \nabla g P_2^{-1}(z)\cdot z_\alpha^\perp)\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}}_{(15)} \\
&\underbrace{-Q^3\left(|BR|^2 + \frac{c^2|z_\alpha|^2}{Q^4} + 2c\frac{BR\cdot z_\alpha}{Q^2} - \frac{\varphi^2}{Q^4}\right)\nabla Q(z)\cdot z_\alpha^\perp \frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}}_{(16)} \\
&+ (Q^2 BR)_\alpha \cdot z_\alpha^\perp \frac{z_{\alpha t}\cdot z_\alpha^\perp}{|z_\alpha|^3} + \underbrace{(Q\omega + 2QBR\cdot z_\alpha)\nabla Q(z)\cdot z_\alpha^\perp \frac{z_{\alpha t}\cdot z_\alpha^\perp}{|z_\alpha|^3}}_{(18)} \\
&\underbrace{-\left(\frac{Q^3(|BR|^2)_\alpha}{|z_\alpha|} + \frac{(c^2|z_\alpha|)_\alpha}{Q} + \left(2cBR\cdot\frac{z_\alpha}{|z_\alpha|}\right)_\alpha Q\right)Q_\alpha}_{(17)}.
\end{aligned}$$

We add and subtract terms in order to find the R-T condition. We recall here that

$$\begin{aligned}
\sigma \equiv& \left(BR_t + \frac{\varphi}{|z_\alpha|}BR_\alpha\right)\cdot z_\alpha^\perp + \frac{\omega}{2|z_\alpha|^2}\left(z_{\alpha t} + \frac{\varphi}{|z_\alpha|}z_{\alpha\alpha}\right)\cdot z_\alpha^\perp \\
&+ Q\left|BR + \frac{\omega}{2|z_\alpha|^2}z_\alpha\right|^2 \nabla Q(z)\cdot z_\alpha^\perp + \nabla g P_2^{-1}(z)\cdot z_\alpha^\perp.
\end{aligned}$$

Then, we find

$$
\begin{aligned}
\varphi_{\alpha t} =& \text{NICE } - \frac{\varphi}{|z_\alpha|}\varphi_{\alpha\alpha} \\
& \underbrace{\begin{aligned}&- Q^2\left(\left(BR_t + \frac{\varphi}{|z_\alpha|}BR_\alpha\right)\cdot z_\alpha^\perp + \frac{\omega}{2|z_\alpha|^2}\left(z_{\alpha t} + \frac{\varphi}{|z_\alpha|}z_{\alpha\alpha}\right)\cdot z_\alpha^\perp + \nabla g P_2^{-1}(z)\cdot z_\alpha^\perp\right)\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3} \\
&+ (Q^2BR)_\alpha \cdot z_\alpha^\perp \frac{z_{\alpha t}\cdot z_\alpha^\perp}{|z_\alpha|^3} + Q^2\left(\frac{\varphi}{|z_\alpha|}BR_\alpha\cdot z_\alpha^\perp + \frac{\omega}{2|z_\alpha|^2}\left(z_{\alpha t} + \frac{\varphi}{|z_\alpha|}z_{\alpha\alpha}\right)\cdot z_\alpha^\perp\right)\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}\end{aligned}}_{(19)} \\
& - Q^3\left(|BR|^2 + \frac{c^2|z_\alpha|^2}{Q^4} + 2c\frac{BR\cdot z_\alpha}{Q^2} - \frac{\varphi^2}{Q^4}\right)\nabla Q(z)\cdot z_\alpha^\perp \frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3} \\
& + (Q\omega + 2QBR\cdot z_\alpha)\nabla Q(z)\cdot z_\alpha^\perp \frac{z_{\alpha t}\cdot z_\alpha^\perp}{|z_\alpha|^3} \\
& - \left(\frac{Q^3(|BR|^2)_\alpha}{|z_\alpha|} + \frac{(c^2|z_\alpha|)_\alpha}{Q} + \left(2cBR\cdot\frac{z_\alpha}{|z_\alpha|}\right)_\alpha Q\right)Q_\alpha.
\end{aligned}
$$

Line (19) can be written as

$$
\begin{aligned}
(19) =& (Q^2BR)_\alpha\cdot z_\alpha^\perp\frac{z_{\alpha t}\cdot z_\alpha^\perp}{|z_\alpha|^3} + Q^2BR_\alpha\cdot z_\alpha^\perp\frac{\varphi}{|z_\alpha|}\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3} + \frac{Q^2\omega}{2|z_\alpha|^2}\left(z_{\alpha t} + \frac{\varphi}{|z_\alpha|}z_{\alpha\alpha}\right)\cdot z_\alpha^\perp\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3} \\
=& (Q^2BR)_\alpha\cdot z_\alpha^\perp\frac{z_{\alpha t}\cdot z_\alpha^\perp}{|z_\alpha|^3} + (Q^2BR)_\alpha\cdot z_\alpha^\perp\frac{\varphi}{|z_\alpha|}\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3} + \frac{Q^2\omega}{2|z_\alpha|^2}\left(z_{\alpha t}\cdot z_\alpha^\perp + \frac{\varphi}{|z_\alpha|}z_{\alpha\alpha}\cdot z_\alpha^\perp\right)\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3} \\
& - 2QQ_\alpha BR\cdot z_\alpha^\perp\frac{\varphi}{|z_\alpha|}\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3} \\
=& (Q^2BR)_\alpha\cdot z_\alpha^\perp\frac{1}{|z_\alpha|^3}\left(z_{\alpha t}\cdot z_\alpha^\perp + \frac{\varphi}{|z_\alpha|}z_{\alpha\alpha}\cdot z_\alpha^\perp\right) + \frac{Q^2\omega}{2|z_\alpha|^2}\frac{1}{|z_\alpha|^3}\left(z_{\alpha t}\cdot z_\alpha^\perp + \frac{\varphi}{|z_\alpha|}z_{\alpha\alpha}\cdot z_\alpha^\perp\right)z_{\alpha\alpha}\cdot z_\alpha^\perp \\
& - 2QQ_\alpha BR\cdot z_\alpha^\perp\frac{\varphi}{|z_\alpha|}\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3} \\
=& \frac{1}{|z_\alpha|^3}\left(z_{\alpha t}\cdot z_\alpha^\perp + \frac{\varphi}{|z_\alpha|}z_{\alpha\alpha}\cdot z_\alpha^\perp\right)\left((Q^2BR)_\alpha\cdot z_\alpha^\perp + \frac{Q^2\omega}{2|z_\alpha|^2}z_{\alpha\alpha}\cdot z_\alpha^\perp\right) \\
& - 2QQ_\alpha BR\cdot z_\alpha^\perp\frac{\varphi}{|z_\alpha|}\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}.
\end{aligned}
$$

We expand $z_{\alpha t}$ to find

$$
\begin{aligned}
(19) =& \frac{1}{|z_\alpha|^3}\left((Q^2BR)_\alpha\cdot z_\alpha^\perp + \frac{Q^2\omega}{2|z_\alpha|^2}z_{\alpha\alpha}\cdot z_\alpha^\perp\right)^2 \\
& - 2QQ_\alpha BR\cdot z_\alpha^\perp\frac{\varphi}{|z_\alpha|}\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}.
\end{aligned}
$$

We denote

$$
D(\alpha) = (Q^2BR)_\alpha\cdot z_\alpha^\perp + \frac{Q^2\omega}{2|z_\alpha|^2}z_{\alpha\alpha}\cdot z_\alpha^\perp. \tag{2.89}
$$

We claim that

$$D(\alpha) = \mathrm{AN}(\alpha) + |z_\alpha| H(\partial_\alpha \varphi) \tag{2.90}$$

where

$$\|\mathrm{AN}\|_{H^3} \leq C E^p(t). \tag{2.91}$$

That means

$$(D(\alpha))^2 = \mathrm{NICE}.$$

Thus

$$(19) = \mathrm{NICE} - 2QQ_\alpha BR \cdot z_\alpha^\perp \frac{\varphi}{|z_\alpha|} \frac{z_{\alpha\alpha} \cdot z_\alpha^\perp}{|z_\alpha|^3}.$$

We write

$$\begin{aligned} D(\alpha) = & \underbrace{2QQ_\alpha BR \cdot z_\alpha^\perp}_{\text{part of AN, at the level of } z_\alpha} + \underbrace{Q^2 \frac{1}{2\pi} PV \int \frac{(z_\alpha - z_\alpha') \cdot z_\alpha}{|z - z'|^2} \omega' d\beta}_{\text{part of AN, we use (2.82)}} \\ & \underbrace{- Q^2 \frac{1}{\pi} PV \int \frac{(z - z') \cdot z_\alpha}{|z - z'|^4} (z - z') \cdot (z_\alpha - z_\alpha') \omega' d\beta}_{\text{part of AN, we use (2.80) and (2.81)}} \\ & + \underbrace{Q^2 BR(z, \omega_\alpha) \cdot z_\alpha^\perp}_{\mathrm{AN} + \frac{Q^2}{2} H(\omega_\alpha)} + \frac{Q^2 \omega}{2|z_\alpha|^2} z_{\alpha\alpha} \cdot z_\alpha^\perp. \end{aligned}$$

Therefore

$$\begin{aligned} D(\alpha) &= \mathrm{AN} + |z_\alpha| Q^2 H\left(\left(\frac{\omega}{2|z_\alpha|}\right)_\alpha\right) + \frac{Q^2 \omega}{2|z_\alpha|^2} z_{\alpha\alpha} \cdot z_\alpha^\perp \\ &= \mathrm{AN} + |z_\alpha| H\left(\left(\frac{Q^2 \omega}{2|z_\alpha|}\right)_\alpha\right) + \frac{Q^2 \omega}{2|z_\alpha|^2} z_{\alpha\alpha} \cdot z_\alpha^\perp \\ &= \mathrm{AN} + |z_\alpha| H(\partial_\alpha \varphi) + H\left((c|z_\alpha|^2)_\alpha\right) + \frac{Q^2 \omega}{2|z_\alpha|^2} z_{\alpha\alpha} \cdot z_\alpha^\perp \\ &= \mathrm{AN} + |z_\alpha| H(\varphi_\alpha) - H\left((Q^2 BR)_\alpha \cdot z_\alpha\right) + \frac{Q^2 \omega}{2|z_\alpha|^2} z_{\alpha\alpha} \cdot z_\alpha^\perp. \end{aligned}$$

We have

$$\begin{aligned} (Q^2 BR)_\alpha \cdot z_\alpha = & \underbrace{2QQ_\alpha BR \cdot z_\alpha}_{\mathrm{AN}} + Q^2 \frac{1}{2\pi} PV \int \frac{(z_\alpha - z_\alpha')^\perp \cdot z_\alpha}{|z - z'|^2} \omega' d\beta \\ & \underbrace{- Q^2 \frac{1}{\pi} PV \int \frac{(z - z')^\perp \cdot z_\alpha}{|z - z'|^4} (z - z') \cdot (z_\alpha - z_\alpha') \omega' d\beta}_{\text{AN, we use that } (z-z')^\perp \cdot z_\alpha = (z - z' - \beta z_\alpha)^\perp \cdot z_\alpha} \\ & + \underbrace{Q^2 \frac{1}{2\pi} PV \int \frac{(z - z')^\perp \cdot z_\alpha}{|z - z'|^2} \omega_\alpha' d\beta}_{\text{AN, we use that } (z-z')^\perp \cdot z_\alpha = (z - z' - \beta z_\alpha)^\perp \cdot z_\alpha}. \end{aligned}$$

For the second term on the right one finds

$$\begin{aligned}
\partial_\alpha^3\Big(\frac{Q^2}{2\pi}PV\int\frac{(z_\alpha-z'_\alpha)^\perp\cdot z_\alpha}{|z-z'|^2}\omega' d\beta\Big) &= \frac{Q^2}{2\pi}PV\int\frac{(\partial_\alpha^4 z_\alpha-\partial_\alpha^4 z')^\perp\cdot z_\alpha}{|z-z'|^2}\omega' d\beta\\
&+\frac{Q^2}{2\pi}PV\int\frac{(z_\alpha-z'_\alpha)^\perp}{|z-z'|^2}\omega' d\beta\cdot\partial_\alpha^4 z+\frac{Q^2}{2\pi}PV\int\frac{(z_\alpha-z'_\alpha)^\perp\cdot z_\alpha}{|z-z'|^2}\partial_\alpha^3\omega' d\beta\\
&-\frac{Q^2}{\pi}PV\int\frac{(z_\alpha-z'_\alpha)^\perp\cdot z_\alpha}{|z-z'|^4}(z-z')\cdot(\partial_\alpha^3 z-\partial_\alpha^3 z')\omega' d\beta+\text{l.o.t.},
\end{aligned}$$

where in l.o.t. we gather the terms of lower order. Then, all the terms above can be estimated in L^2 but the first one on the right. That is equal to

$$\frac{1}{2}H\left(Q^2\frac{\partial_\alpha^5 z^\perp\cdot z_\alpha}{|z_\alpha|^2}\omega\right)$$

plus a commutator which can be estimated in L^2. This means that

$$(Q^2BR)_\alpha\cdot z_\alpha=\text{AN}+\frac{1}{2}H\left(Q^2\frac{z_{\alpha\alpha}^\perp\cdot z_\alpha}{|z_\alpha|^2}\omega\right).$$

Taking Hilbert transforms:

$$-H\left((Q^2BR)_\alpha\cdot z_\alpha\right)=\text{AN}-\frac{1}{2}H^2\left(Q^2\frac{z_{\alpha\alpha}^\perp\cdot z_\alpha}{|z_\alpha|^2}\omega\right)=\text{AN}+\frac{1}{2}Q^2\frac{z_{\alpha\alpha}^\perp\cdot z_\alpha}{|z_\alpha|^2}\omega.$$

Using that $z_{\alpha\alpha}^\perp\cdot z_\alpha=-z_{\alpha\alpha}\cdot z_\alpha^\perp$ we complete the proof of (2.90). Thus (19) yields

$$\begin{aligned}
\varphi_{\alpha t}=&\text{NICE}-\frac{\varphi}{|z_\alpha|}\varphi_{\alpha\alpha}\\
&-Q^2\left(\left(BR_t+\frac{\varphi}{|z_\alpha|}BR_\alpha\right)\cdot z_\alpha^\perp+\frac{\omega}{2|z_\alpha|^2}\left(z_{\alpha t}+\frac{\varphi}{|z_\alpha|}z_{\alpha\alpha}\right)\cdot z_\alpha^\perp+\nabla gP_2^{-1}(z)\cdot z_\alpha^\perp\right)\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}\\
&-Q^3\left(|BR|^2+\frac{c^2|z_\alpha|^2}{Q^4}+2c\frac{BR\cdot z_\alpha}{Q^2}-\frac{\varphi^2}{Q^4}\right)\nabla Q(z)\cdot z_\alpha^\perp\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}\\
&+(Q\omega+2QBR\cdot z_\alpha)\nabla Q(z)\cdot z_\alpha^\perp\frac{z_{\alpha t}\cdot z_\alpha^\perp}{|z_\alpha|^3}\\
&\underbrace{-\left(\frac{Q^3(|BR|^2)_\alpha}{|z_\alpha|}+\frac{(c^2|z_\alpha|)_\alpha}{Q}+\left(2cBR\cdot\frac{z_\alpha}{|z_\alpha|}\right)_\alpha Q\right)Q_\alpha}_{(20)}\\
&\underbrace{-2QQ_\alpha BR\cdot z_\alpha^\perp\frac{\varphi}{|z_\alpha|}\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}}_{(21)}.
\end{aligned}$$

For (20) we write

$$|z_t|^2 = Q^4|BR|^2 + c^2|z_\alpha|^2 + 2Q^2cBR\cdot z_\alpha$$
$$\Rightarrow \frac{|z_t|^2}{Q|z_\alpha|} = \frac{Q^3|BR|^2}{|z_\alpha|} + \frac{c^2|z_\alpha|}{Q} + 2QcBR\cdot\frac{z_\alpha}{|z_\alpha|}.$$

Now

$$(20) = \text{NICE } -\frac{(|z_t|^2)_\alpha}{Q|z_\alpha|}Q_\alpha,$$

which means

$$(20)+(21) = \text{NICE} - \frac{(|z_t|^2)_\alpha}{Q|z_\alpha|}Q_\alpha - 2QQ_\alpha BR\cdot z_\alpha^\perp\frac{\varphi}{|z_\alpha|}\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}.$$

We write

$$\begin{aligned} z_{\alpha t} &= \underbrace{(z_{\alpha t}\cdot z_\alpha)}_{\text{only depends on } t}\frac{z_\alpha}{|z_\alpha|^2} + (z_{\alpha t}\cdot z_\alpha^\perp)\frac{z_\alpha^\perp}{|z_\alpha|^2} \\ &= \underbrace{B(t)}_{\text{See (2.65)}}z_\alpha + ((Q^2BR)_\alpha\cdot z_\alpha^\perp + cz_{\alpha\alpha}\cdot z_\alpha^\perp)\frac{z_\alpha^\perp}{|z_\alpha|^2} \\ &= B(t)z_\alpha + \underbrace{D(\alpha)}_{\text{as in (5.14)}}\frac{z_\alpha^\perp}{|z_\alpha|^2} - \frac{\varphi}{|z_\alpha|}z_{\alpha\alpha}\cdot z_\alpha^\perp\frac{z_\alpha^\perp}{|z_\alpha|^2}. \end{aligned}$$

Writing $z_t = Q^2BR + cz_\alpha$ we compute

$$\begin{aligned} z_{\alpha t}\cdot z_t = \underbrace{Q^2BR\cdot z_\alpha B(t)}_{\text{NICE}} + \underbrace{DQ^2BR\cdot\frac{z_\alpha^\perp}{|z_\alpha|^2}}_{\text{NICE because } D \text{ is nice}} \\ -\frac{\varphi}{|z_\alpha|}z_{\alpha\alpha}\cdot z_\alpha^\perp Q^2BR\cdot\frac{z_\alpha^\perp}{|z_\alpha|^2} + \underbrace{cB(t)|z_\alpha|^2}_{\text{NICE}}. \end{aligned}$$

To simplify we write

$$z_{\alpha t}\cdot z_t = \text{NICE} - \frac{\varphi}{|z_\alpha|}z_{\alpha\alpha}\cdot z_\alpha^\perp Q^2BR\cdot\frac{z_\alpha^\perp}{|z_\alpha|^2}.$$

Setting the above formula in the expression of (20)+(21) allows us to find

$$(20)+(21) = \text{NICE }.$$

This yields

$$\begin{aligned}\varphi_{\alpha t} =& \text{NICE } - \frac{\varphi}{|z_\alpha|}\varphi_{\alpha\alpha} \\ &- Q^2\left(\left(BR_t + \frac{\varphi}{|z_\alpha|}BR_\alpha\right) + \frac{\omega}{2|z_\alpha|^2}\left(z_{\alpha t} + \frac{\varphi}{|z_\alpha|}z_{\alpha\alpha}\right) + \nabla g P_2^{-1}(z)\right)\cdot z_\alpha^\perp \frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3} \\ &- Q^3\left(|BR|^2 + \frac{c^2|z_\alpha|^2}{Q^4} + 2c\frac{BR\cdot z_\alpha}{Q^2} - \frac{\varphi^2}{Q^4}\right)\nabla Q(z)\cdot z_\alpha^\perp \frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3} \\ &+ (Q\omega + 2QBR\cdot z_\alpha)\nabla Q(z)\cdot z_\alpha^\perp \frac{z_{\alpha t}\cdot z_\alpha^\perp}{|z_\alpha|^3}.\end{aligned}$$

We now complete the formula for σ in (5.2) to find

$$\begin{aligned}\varphi_{\alpha t} =& \text{NICE} - \frac{\varphi}{|z_\alpha|}\varphi_{\alpha\alpha} - Q^2\sigma\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3} \\ &+ \underbrace{Q^3\left|BR + \frac{\omega}{2|z_\alpha|^2}z_\alpha\right|^2 \nabla Q\cdot z_\alpha^\perp \frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}}_{(22)} \\ &+ \underbrace{Q^3\left(-|BR|^2 - \frac{c^2|z_\alpha|^2}{Q^4} - 2c\frac{BR\cdot z_\alpha}{Q^2} + \frac{\varphi^2}{Q^4}\right)\nabla Q(z)\cdot z_\alpha^\perp \frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}}_{(23)} \\ &+ \underbrace{(Q\omega + 2QBR\cdot z_\alpha)\nabla Q(z)\cdot z_\alpha^\perp \frac{z_{\alpha t}\cdot z_\alpha^\perp}{|z_\alpha|^3}}_{(24)}.\end{aligned}$$

Expanding

$$\frac{\varphi^2}{Q^4} = \frac{\omega^2}{4|z_\alpha|^2} + \frac{c^2|z_\alpha|^2}{Q^4} - \frac{\omega c}{Q^2}$$

we find

$$(22) + (23) = Q^3\left(\frac{\omega^2}{2|z_\alpha|^2} + BR\cdot z_\alpha\frac{\omega}{|z_\alpha|^2} - 2c\frac{BR\cdot z_\alpha}{Q^2} - \frac{\omega c}{Q^2}\right)\nabla Q(z)\cdot z_\alpha^\perp \frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}.$$

Writing

$$z_{\alpha t}\cdot z_\alpha^\perp = (Q^2BR)_\alpha\cdot z_\alpha^\perp + c z_{\alpha\alpha}\cdot z_\alpha^\perp$$

we obtain that

$$\begin{aligned}(24) =& (Q\omega + 2QBR\cdot z_\alpha)\nabla Q(z)\cdot z_\alpha^\perp \frac{(Q^2BR)_\alpha\cdot z_\alpha^\perp}{|z_\alpha|^3} \\ &+ (Q\omega + 2QBR\cdot z_\alpha)\nabla Q(z)\cdot z_\alpha^\perp c\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}.\end{aligned}$$

Thus

$$
\begin{aligned}
(22)+(23)+(24) &= Q^3\left(\frac{\omega^2}{2|z_\alpha|^2}+BR\cdot z_\alpha\frac{\omega}{|z_\alpha|^2}\right)\nabla Q(z)\cdot z_\alpha^\perp\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}\\
&\quad+(Q\omega+2QBR\cdot z_\alpha)\nabla Q(z)\cdot z_\alpha^\perp\frac{(Q^2BR)_\alpha\cdot z_\alpha^\perp}{|z_\alpha|^3}\\
&= Q\nabla Q(z)\cdot z_\alpha^\perp(\omega+2BR\cdot z_\alpha)\left(\frac{Q^2\omega}{2|z_\alpha|^2}\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}+\frac{(Q^2BR)_\alpha\cdot z_\alpha^\perp}{|z_\alpha|^3}\right)\\
&= Q\nabla Q(z)\cdot z_\alpha^\perp(\omega+2BR\cdot z_\alpha)\frac{1}{|z_\alpha|^3}D(\alpha)\\
&= \text{NICE}.
\end{aligned}
$$

Finally, we obtain

$$
\varphi_{\alpha t}=\text{NICE}-\frac{\varphi}{|z_\alpha|}\varphi_{\alpha\alpha}-Q^2\sigma\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}.
$$

□

Corollary 2.4.13 *If we disregard the condition on the H^{k-2} norm for the definition of the NICE terms, imposing only the first condition, then*

$$
\varphi_{\alpha t}=NICE-Q^2\sigma\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}.
$$

2.4.4.11 Higher order derivatives of σ

In this section we deal with the highest order derivative of the R-T function. We show that

Lemma 2.4.14 *Let $z(\alpha,t)$ and $\omega(\alpha,t)$ be a solution of (3.1-3.2). Then, the following identity holds:*

$$
\partial_\alpha^{k-1}(Q^2\sigma)=ANN+|z_\alpha|H(\partial_\alpha^{k-1}\varphi_t)+\varphi H(\partial_\alpha^k\varphi) \tag{2.92}
$$

where ANN satisfies

$$
\|ANN\|_{L^2}\le CE^p(t) \tag{2.93}
$$

for $k\ge 4$, where C and p are constants that depend only on k.

Proof: We show the proof for $k=4$. From now on, if a term f satisfies

$$
\|f\|_{L^2}\le CE^p(t)
$$

we say that this term becomes part of ANN. By abuse of notation we will denote f by ANN. We recall

$$
\begin{aligned}
Q^2\sigma =& Q^2\left(BR_t + \frac{\varphi}{|z_\alpha|}BR_\alpha\right)\cdot z_\alpha^\perp + \frac{Q^2\omega}{2|z_\alpha|^2}\left(z_{\alpha t} + \frac{\varphi}{|z_\alpha|}z_{\alpha\alpha}\right)\cdot z_\alpha^\perp \\
&+ \underbrace{Q^3\left|BR + \frac{\omega}{2|z_\alpha|^2}z_\alpha\right|^2 \nabla Q\cdot z_\alpha^\perp}_{\text{this term is in } H^3 \text{ so its third derivative is in ANN}} + \underbrace{Q^2\nabla g P_2^{-1}(z)\cdot z_\alpha^\perp}_{\text{this term is also in } H^3}.
\end{aligned}
$$

We write

$$
\begin{aligned}
\frac{Q^2\omega}{2|z_\alpha|^2}\left(z_{\alpha t} + \frac{\varphi}{|z_\alpha|}z_{\alpha\alpha}\right)\cdot z_\alpha^\perp &= \frac{Q^2\omega}{2|z_\alpha|^2}\left((Q^2BR)_\alpha\cdot z_\alpha^\perp + cz_{\alpha\alpha}\cdot z_\alpha^\perp + \frac{\varphi}{|z_\alpha|}z_{\alpha\alpha}\cdot z_\alpha^\perp\right) \\
&= \frac{Q^2\omega}{2|z_\alpha|^2}\left((Q^2BR)_\alpha\cdot z_\alpha^\perp + \left(c + \frac{\varphi}{|z_\alpha|}\right)z_{\alpha\alpha}\cdot z_\alpha^\perp\right) \\
&= \frac{Q^2\omega}{2|z_\alpha|^2}\left((Q^2BR)_\alpha\cdot z_\alpha^\perp + \frac{Q^2\omega}{2|z_\alpha|^2}z_{\alpha\alpha}\cdot z_\alpha^\perp\right) \\
&= \frac{Q^2\omega}{2|z_\alpha|^2}D(\alpha).
\end{aligned}
$$

Above we use (5.14) and (2.90) to find

$$
\frac{Q^2\omega}{2|z_\alpha|^2}\left(z_{\alpha t} + \frac{\varphi}{|z_\alpha|}z_{\alpha\alpha}\right)\cdot z_\alpha^\perp = \mathrm{AN} + \frac{Q^2\omega}{2|z_\alpha|}H(\varphi_\alpha), \tag{2.94}
$$

where AN is as in (2.91). The remaining terms in $Q^2\sigma$ are

$$
L = Q^2BR_t\cdot z_\alpha^\perp + \frac{Q^2\varphi}{|z_\alpha|}BR_\alpha\cdot z_\alpha^\perp.
$$

We take 3 derivatives and consider the most dangerous characters:

$$
\partial_\alpha^3(L) = M_1 + M_2 + M_3 + \mathrm{ANN},
$$

where

$$
M_1 = Q^2BR(z,\partial_\alpha^3\omega_t)\cdot z_\alpha^\perp + \frac{Q^2\varphi}{|z_\alpha|}BR(z,\partial_\alpha^4\omega)\cdot z_\alpha^\perp,
$$

$$
\begin{aligned}
M_2 =& Q^2\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\partial_\alpha^3 z_t - \partial_\alpha^3 z_t')\cdot z_\alpha}{|z-z'|^2}\omega' d\beta \\
&+ \frac{Q^2\varphi}{|z_\alpha|}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\partial_\alpha^4 z - \partial_\alpha^4 z')\cdot z_\alpha}{|z-z'|^2}\omega' d\beta,
\end{aligned}
$$

$$
\begin{aligned}
M_3 =& -\frac{Q^2}{\pi}\int_{-\pi}^{\pi}\frac{(z-z')\cdot z_\alpha}{|z-z'|^4}(z-z')\cdot(\partial_\alpha^3 z_t - \partial_\alpha^3 z_t')\omega' d\beta \\
&- \frac{Q^2\varphi}{|z_\alpha|\pi}\int_{-\pi}^{\pi}\frac{(z-z')\cdot z_\alpha}{|z-z'|^4}(z-z')\cdot(\partial_\alpha^4 z - \partial_\alpha^4 z')\omega' d\beta.
\end{aligned}
$$

Here we point out that in order to deal with BR_t in the less singular terms we proceed using estimate (2.74). In M_2 we find

$$M_2 = \frac{Q^2\omega}{2|z_\alpha|^2}\Lambda(\partial_\alpha^3 z_t \cdot z_\alpha) + \frac{Q^2\varphi\omega}{2|z_\alpha|^3}\Lambda(\partial_\alpha^4 z \cdot z_\alpha) + \text{ANN}.$$

For the second term we use the usual trick

$$\partial_\alpha^4 z \cdot z_\alpha = -3\partial_\alpha^3 z \cdot z_{\alpha\alpha}.$$

For the first term we recall that

$$\begin{aligned}
&|z_\alpha|^2 = A(t) \Rightarrow z_\alpha \cdot z_{\alpha t} = \frac{1}{2}A'(t) \Rightarrow (z_\alpha \cdot z_{\alpha t})_\alpha = 0\\
&\Rightarrow z_{\alpha\alpha} \cdot z_{\alpha t} + z_\alpha \cdot z_{\alpha\alpha t} = 0 \Rightarrow z_{\alpha\alpha\alpha} \cdot z_{\alpha t} + 2z_{\alpha\alpha} \cdot z_{\alpha\alpha t} + z_\alpha \cdot z_{\alpha\alpha\alpha t} = 0\\
&\Rightarrow z_\alpha \cdot z_{\alpha\alpha\alpha t} = -2z_{\alpha\alpha} \cdot z_{\alpha\alpha t} - z_{\alpha\alpha\alpha} \cdot z_{\alpha t}.
\end{aligned}$$

This allows us to control M_2. For M_3 we find

$$M_3 = -\frac{Q^2\omega}{|z_\alpha|^2}\Lambda(z_\alpha \cdot \partial_\alpha^3 z_t) - \frac{Q^2\varphi\omega}{|z_\alpha|^3}\Lambda(z_\alpha \cdot \partial_\alpha^4 z) + \text{ANN}$$

so it can be estimated as M_2. There remains M_1. Using that $(z-z')^\perp \cdot z_\alpha^\perp = (z-z') \cdot z_\alpha$ we find

$$M_1 = \frac{Q^2}{2}H(\partial_\alpha^3 \omega_t) + \frac{Q^2\varphi}{2|z_\alpha|}H(\partial_\alpha^4 \omega) + \text{ANN}. \tag{2.95}$$

We compute

$$\begin{aligned}
\frac{Q^2}{2}H(\partial_\alpha^3\omega_t) &= H\left(\partial_\alpha^3\left(\frac{Q^2\omega}{2}\right)_t\right) + \text{ANN}\\
&= H(\partial_\alpha^3(|z_\alpha|\varphi)_t) + H(\partial_\alpha^3(|z_\alpha|c)_t) + \text{ANN}\\
&= |z_\alpha|H(\partial_\alpha^3\varphi_t) + H(\partial_\alpha^2\partial_t(-(Q^2BR)_\alpha \cdot z_\alpha)) + \text{ANN}.
\end{aligned} \tag{2.96}$$

We compute the most singular term in

$$\begin{aligned}
\partial_\alpha^2\partial_t(-(Q^2BR)_\alpha \cdot z_\alpha) = &-\frac{Q^2}{2\pi}\int_{-\pi}^{\pi}\frac{(\partial_\alpha^3 z_t - \partial_\alpha^3 z_t')^\perp \cdot z_\alpha}{|z-z'|^2}\omega' d\beta\\
&+\underbrace{\frac{Q^2}{\pi}\int_{-\pi}^{\pi}\frac{(z-z')^\perp \cdot z_\alpha}{|z-z'|^4}(z-z')\cdot(\partial_\alpha^3 z_t - \partial_\alpha^3 z_t')\omega' d\beta}_{\text{extra cancelation in } (z-z')^\perp\cdot z_\alpha = (z-z'-z_\alpha\beta)^\perp\cdot z_\alpha}\\
&\underbrace{-\frac{Q^2}{2\pi}PV\int_{-\pi}^{\pi}\frac{(z-z')^\perp \cdot z_\alpha}{|z-z'|^2}\partial_\alpha^3\omega_t' d\beta}_{\text{extra cancelation as above}} + \text{ANN}.
\end{aligned}$$

This shows that

$$\partial_\alpha^2\partial_t(-(Q^2BR)_\alpha\cdot z_\alpha)=-\frac{Q^2\omega}{2|z_\alpha|^2}\Lambda(\partial_\alpha^3 z_t^\perp\cdot z_\alpha)+\text{ANN}.$$

That gives

$$\partial_\alpha^2\partial_t(-(Q^2BR)_\alpha\cdot z_\alpha)=-\Lambda\left(\frac{Q^2\omega}{2|z_\alpha|^2}\partial_\alpha^3 z_t^\perp\cdot z_\alpha\right)+\text{ANN},$$

which implies

$$H(\partial_\alpha^2\partial_t(-(Q^2BR)_\alpha\cdot z_\alpha))=\partial_\alpha\left(\frac{Q^2\omega}{2|z_\alpha|^2}\partial_\alpha^3 z_t^\perp\cdot z_\alpha\right)+\text{ANN}=-\frac{Q^2\omega}{2|z_\alpha|^2}\partial_\alpha\left(\partial_\alpha^3 z_t\cdot z_\alpha^\perp\right)+\text{ANN}.$$

Plugging the above formula in (5.17) we find that

$$\begin{aligned}\frac{Q^2}{2}H(\partial_\alpha^3\omega_t)&=|z_\alpha|H(\partial_\alpha^3\varphi_t)-\frac{Q^2\omega}{2|z_\alpha|^2}\partial_\alpha\left(\partial_\alpha^3 z_t\cdot z_\alpha^\perp\right)+\text{ANN}\\&=|z_\alpha|H(\partial_\alpha^3\varphi_t)-\frac{Q^2\omega}{2|z_\alpha|^2}\partial_\alpha\left(\partial_\alpha^3(Q^2BR)\cdot z_\alpha^\perp\right)-\frac{Q^2\omega}{2|z_\alpha|^2}\partial_\alpha\left(c\partial_\alpha^4 z\cdot z_\alpha^\perp\right)\\&\quad+\text{ANN}.\end{aligned}$$

As we did before, we expand $\partial_\alpha(\partial_\alpha^3(Q^2BR)\cdot z_\alpha^\perp)$ to find

$$\begin{aligned}\partial_\alpha(\partial_\alpha^3(Q^2BR)\cdot z_\alpha^\perp)=&2Q\nabla Q(z)\cdot\partial_\alpha^4 zBR\cdot z_\alpha^\perp+\frac{Q^2}{2\pi}PV\int\frac{(\partial_\alpha^4 z-\partial_\alpha^4 z')\cdot z_\alpha}{|z-z'|^2}\omega' d\beta\\&-\frac{Q^2}{\pi}PV\int\frac{(z-z')\cdot z_\alpha}{|z-z'|^4}(z-z')\cdot(\partial_\alpha^4 z-\partial_\alpha^4 z')\omega' d\beta\\&+\frac{Q^2}{2\pi}PV\int\frac{(z-z')\cdot z_\alpha}{|z-z'|^2}\partial_\alpha^4\omega' d\beta+\text{ANN}.\end{aligned}$$

Therefore, we can use (2.80),(2.81) and (2.81) to show that the most dangerous term is given by $Q^2\frac{1}{2}H(\partial_\alpha^4\omega)$. It implies

$$\partial_\alpha(\partial_\alpha^3(Q^2BR)\cdot z_\alpha^\perp)=Q^2\frac{1}{2}H(\partial_\alpha^4\omega)+\text{ANN}$$

and therefore

$$\frac{Q^2}{2}H(\partial_\alpha^3\omega_t)=|z_\alpha|H(\partial_\alpha^3\varphi_t)-\frac{Q^2\omega}{2|z_\alpha|^2}\frac{Q^2}{2}H(\partial_\alpha^4\omega)-\frac{Q^2\omega}{2|z_\alpha|^2}c\partial_\alpha\left(\partial_\alpha^4 z\cdot z_\alpha^\perp\right)+\text{ANN}.$$

We use the above formula and expand φ to find

$$\begin{aligned}
M_1 &= \frac{Q^2}{2} H(\partial_\alpha^3 \omega_t) + \frac{Q^2 \varphi}{2|z_\alpha|} H(\partial_\alpha^4 \omega) + \text{ANN} \\
&= |z_\alpha| H(\partial_\alpha^3 \varphi_t) - \frac{Q^2}{2} c H(\partial_\alpha^4 \omega) - \frac{Q^2 \omega}{2|z_\alpha|^2} c \partial_\alpha \left(\partial_\alpha^4 z \cdot z_\alpha^\perp \right) + \text{ANN} \\
&= |z_\alpha| H(\partial_\alpha^3 \varphi_t) - c|z_\alpha| H \left(\partial_\alpha^4 \left(\frac{Q^2 \omega}{2|z_\alpha|} \right) \right) - \frac{Q^2 \omega}{2|z_\alpha|^2} c \partial_\alpha \left(\partial_\alpha^4 z \cdot z_\alpha^\perp \right) + \text{ANN} \\
&= |z_\alpha| H(\partial_\alpha^3 \varphi_t) - c|z_\alpha| H \left(\partial_\alpha^4 \varphi \right) - c|z_\alpha| H(\partial_\alpha^4 (c|z_\alpha|)) - \frac{Q^2 \omega}{2|z_\alpha|^2} c \partial_\alpha \left(\partial_\alpha^4 z \cdot z_\alpha^\perp \right) + \text{ANN}.
\end{aligned}$$

We will show that

$$-c|z_\alpha| H(\partial_\alpha^4 (c|z_\alpha|)) - \frac{Q^2 \omega}{2|z_\alpha|^2} c \partial_\alpha \left(\partial_\alpha^4 z \cdot z_\alpha^\perp \right) = \text{ANN}. \tag{2.97}$$

It yields

$$\frac{Q^2}{2} H(\partial_\alpha^3 \omega_t) + \frac{Q^2 \varphi}{2|z_\alpha|} H(\partial_\alpha^4 \omega) = |z_\alpha| H(\partial_\alpha^3 \varphi_t) - c|z_\alpha| H \left(\partial_\alpha^4 \varphi \right) + \text{ANN}$$

that together with (2.94) allows us to obtain (2.92). We have

$$\begin{aligned}
-c|z_\alpha| H(\partial_\alpha^4 (c|z_\alpha|)) - \frac{Q^2 \omega}{2|z_\alpha|^2} c \partial_\alpha \left(\partial_\alpha^4 z \cdot z_\alpha^\perp \right) &= -c H(\partial_\alpha^4 (c|z_\alpha|^2)) - \frac{Q^2 \omega}{2|z_\alpha|^2} c \partial_\alpha \left(\partial_\alpha^4 z \cdot z_\alpha^\perp \right) \\
&= c H(\partial_\alpha^3 ((Q^2 BR)_\alpha \cdot z_\alpha)) - \frac{Q^2 \omega}{2|z_\alpha|^2} c \partial_\alpha \left(\partial_\alpha^4 z \cdot z_\alpha^\perp \right).
\end{aligned}$$

We repeat the calculation for dealing with the most dangerous terms in

$$\partial_\alpha^3 ((Q^2 BR)_\alpha \cdot z_\alpha) = \Lambda \left(\partial_\alpha^4 z^\perp \cdot z_\alpha \frac{\omega Q^2}{2|z_\alpha|^2} \right) + \text{ANN}.$$

We recognized as before terms in ANN using that $(z - z')^\perp \cdot z_\alpha$ gives an extra cancellation. We find that

$$\begin{aligned}
c H(\partial_\alpha^3 ((Q^2 BR)_\alpha \cdot z_\alpha)) &- \frac{Q^2 \omega}{2|z_\alpha|^2} c \partial_\alpha \left(\partial_\alpha^4 z \cdot z_\alpha^\perp \right) \\
&= c H \left(\Lambda \left(\partial_\alpha^4 z^\perp \cdot z_\alpha \frac{\omega Q^2}{2|z_\alpha|^2} \right) \right) - \frac{Q^2 \omega}{2|z_\alpha|^2} c \partial_\alpha \left(\partial_\alpha^4 z \cdot z_\alpha^\perp \right) + \text{ANN} \\
&= -c \partial_\alpha \left(\partial_\alpha^4 z^\perp \cdot z_\alpha \frac{\omega Q^2}{2|z_\alpha|^2} \right) - \frac{Q^2 \omega}{2|z_\alpha|^2} c \partial_\alpha \left(\partial_\alpha^4 z \cdot z_\alpha^\perp \right) + \text{ANN}.
\end{aligned}$$

Using that $\partial_\alpha^4 z^\perp \cdot z_\alpha = -\partial_\alpha^4 z \cdot z_\alpha^\perp$ we are done proving (2.97). $\square$

2.4.4.12 Energy estimates for φ

In this section we prove the following result.

Lemma 2.4.15 *Let $z(\alpha,t)$ and $\omega(\alpha,t)$ be a solution of (3.1-3.2). Then, the following a priori estimate holds:*

$$\frac{d}{dt}\|\varphi\|^2_{H^{k-\frac{1}{2}}}(t) \leq -S(t) + CE^p(t) \tag{2.98}$$

for $k \geq 4$, where C and p are constants that depend only on k.

Proof: We shall present the details in the case $k=4$, leaving the other cases to the reader.

Using the estimates obtained before one has

$$\frac{d}{dt}\|\varphi\|^2_{L^2}(t) \leq CE^p(t).$$

Developing the derivative using Lemma 2.4.12, we get that:

$$\begin{aligned}\frac{d}{dt}\|\Lambda^{1/2}(\partial^3_\alpha\varphi)\|^2_{L^2}(t) =& 2\int_{\mathbb{T}}\Lambda(\partial^3_\alpha\varphi)\partial^3_\alpha\varphi_t d\alpha \\ =& I_1+I_2+I_3,\end{aligned} \tag{2.99}$$

where

$$I_1 = 2\int_{\mathbb{T}}\Lambda(\partial^3_\alpha\varphi)\partial^2_\alpha(\text{NICE})d\alpha, \qquad I_2 = -2\int_{\mathbb{T}}\Lambda(\partial^3_\alpha\varphi)\partial^2_\alpha(\frac{\varphi}{|z_\alpha|}\varphi_{\alpha\alpha})d\alpha,$$

$$I_3 = -2\int_{\mathbb{T}}\Lambda(\partial^3_\alpha\varphi)\partial^2_\alpha(Q^2\sigma\frac{z_{\alpha\alpha}\cdot z^\perp_\alpha}{|z_\alpha|^3})d\alpha.$$

We use (2.85) to control I_1. The most singular term in I_2 is the one given by

$$\begin{aligned}-2\int_{\mathbb{T}}\frac{1}{|z_\alpha|}\Lambda(\partial^3_\alpha\varphi)\partial^4_\alpha\varphi\varphi d\alpha =& 2\int_{\mathbb{T}}\frac{1}{|z_\alpha|}\Lambda^{1/2}(\partial^3_\alpha\varphi)\left[\varphi\Lambda^{1/2}(\partial^4_\alpha\varphi)-\Lambda^{1/2}(\varphi\partial^4_\alpha\varphi)\right]d\alpha \\ &+\int_{\mathbb{T}}\frac{1}{|z_\alpha|}\partial_\alpha\varphi|\Lambda^{1/2}(\partial^3_\alpha\varphi)|^2 d\alpha.\end{aligned}$$

Using the commutator estimate

$$\|g\Lambda^{1/2}(\partial_\alpha f)-\Lambda^{1/2}(g\partial_\alpha f)\|_{L^2} \leq \|g\|_{C^2}\|f\|_{H^{1/2}} \tag{2.100}$$

we can bound I_2. In I_3 we split further considering the most singular terms

$$J_1 = -2\int_{\mathbb{T}}\Lambda(\partial^3_\alpha\varphi)Q^2\sigma z_{\alpha\alpha}\cdot\partial^2_\alpha(\frac{z^\perp_\alpha}{|z_\alpha|^3})d\alpha,$$

$$J_2 = -2\int_{\mathbb{T}}\Lambda(\partial^3_\alpha\varphi)Q^2\sigma\frac{\partial^4_\alpha z\cdot z^\perp_\alpha}{|z_\alpha|^3}d\alpha,$$

$$J_3 = -2\int_{\mathbb{T}} \Lambda(\partial_\alpha^3\varphi)\partial_\alpha^2(Q^2\sigma)\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}d\alpha.$$

The term J_1 can be estimated as before. Recalling (2.79) we see that $J_2 = -S(t)$. It remains to control J_3 in order to find (2.98).

We decompose $J_3 = K_1 + K_2$ where

$$K_1 = 2\int_{\mathbb{T}} H(\partial_\alpha^3\varphi)\partial_\alpha^2(Q^2\sigma)\partial_\alpha(\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3})d\alpha$$

and

$$K_2 = 2\int_{\mathbb{T}} H(\partial_\alpha^3\varphi)\partial_\alpha^3(Q^2\sigma)\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}d\alpha.$$

Inequality (2.75) for $k = 4$ allows us to obtain

$$K_1 \leq CE^p(t).$$

To finish the proof we use formula (2.92) for $k = 4$ to find $K_2 = L_1 + L_2 + L_3$ where

$$L_1 = 2\int_{\mathbb{T}} H(\partial_\alpha^3\varphi)\mathrm{ANN}\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}d\alpha,$$

$$L_2 = 2\int_{\mathbb{T}} H(\partial_\alpha^3\varphi)|z_\alpha|H(\partial_\alpha^3\varphi_t)\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}d\alpha,$$

$$L_3 = 2\int_{\mathbb{T}} H(\partial_\alpha^3\varphi)\varphi H(\partial_\alpha^4\varphi)\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}d\alpha.$$

The term L_1 can be easily estimated using (2.93). For L_2 we substitute the expression (2.98) for $\partial_\alpha^3\varphi_t$ to get $L_2 = M_1 + M_2 + M_3$:

$$M_1 = 2\int_{\mathbb{T}} H(\partial_\alpha^3\varphi)|z_\alpha|H(\partial_\alpha^2(\mathrm{NICE}))\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}d\alpha.$$

$$M_2 = -2\int_{\mathbb{T}} H(\partial_\alpha^3\varphi)|z_\alpha|H(\partial_\alpha^2(\frac{\varphi\varphi_{\alpha\alpha}}{|z_\alpha|}))\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}d\alpha.$$

$$M_3 = -2\int_{\mathbb{T}} H(\partial_\alpha^3\varphi)|z_\alpha|H(\partial_\alpha^2(Q^2\sigma\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}))\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}d\alpha.$$

By equation (2.85), M_1 is bounded. M_2 is bounded knowing that we have room for half derivative in the term which is not the third factor. Finally we can bound M_3 in virtue of Lemma 2.4.8. To finish, in L_3 we integrate by parts to find

$$L_3 = -\int_{\mathbb{T}} |H(\partial_\alpha^3\varphi)|^2\partial_\alpha(\varphi\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3})d\alpha \leq CE^p(t)$$

using Sobolev embedding. □

2.4.4.13 Energy estimates for $\frac{|z_\alpha|^2}{m(Q^2\sigma)(t)} + \sum_{l=0}^{4} \frac{1}{m(q^l)(t)}$.

Lemma 2.4.16 *Let $z(\alpha, t)$ and $\omega(\alpha, t)$ be a solution of (3.1-3.2). Then, the following a priori estimate holds:*

$$\frac{d}{dt}\Big(\frac{|z_\alpha|^2}{m(Q^2\sigma)(t)} + \sum_{l=0}^{4} \frac{1}{m(q^l)(t)}\Big) \leq CE^p(t) \tag{2.101}$$

for $k \geq 4$, where C and p are constants that depend only on k.

Proof: Inequalities (2.71) and (2.76) show that $(Q^2\sigma) \in C^1([0,T] \times [-\pi,\pi])$ for some T and therefore $m(Q^2\sigma)(t)$ is a Lipschitz function differentiable almost everywhere by Rademacher's theorem. Let

$$m(Q^2\sigma)(t) = \min_{\alpha\in[-\pi,\pi]} (Q^2\sigma)(\alpha, t) = (Q^2\sigma)(\alpha_t, t).$$

We can calculate the derivative of $m(Q^2\sigma)(t)$, to obtain

$$(m(Q^2\sigma))'(t) = (Q^2\sigma)_t(\alpha_t, t)$$

for almost every t. Then it follows that:

$$\frac{d}{dt}\left(\frac{1}{m(Q^2\sigma)}\right)(t) = -\frac{(Q^2\sigma)_t(\alpha_t, t)}{(m(Q^2\sigma))^2(t)}$$

almost everywhere. By using the previous a priori estimates for the L^∞ bounds, we get to

$$\frac{d}{dt}\left(\frac{|z_\alpha|^2}{m(Q^2\sigma)}\right)(t) \leq CE^p(t).$$

On the other hand, we can apply the same argument to $\frac{1}{m(q^l)(t)}$. Denoting again by α_t the point where the minimum is attained we have that:

$$\frac{d}{dt}\left(\frac{1}{m(q^l)}\right)(t) = -\frac{z_t(\alpha_t, t)\cdot(z(\alpha_t, t) - q^l)}{(m(q^l))^3(t)}$$

which again can be easily bounded and we get (2.101), as desired.

□

2.4.5 Proof of short-time existence (Theorem 3.5.1)

To conclude the proof of the local existence, we shall use the previous a priori estimates. We now introduce a regularized version of the evolution equation which is well-posed for short time independently of the sign condition on $\sigma(\alpha, t)$ at $t = 0$. But for $\sigma(\alpha, 0) > 0$, we shall find a time of existence uniformly in the regularization, allowing us to take the limit.

Now, let $z^{\varepsilon,\delta,\mu}(\alpha,t)$ be a solution of the following system (compare with (2.64)):

$$z_t^{\varepsilon,\delta,\mu}(\alpha,t)=\phi_\delta*\phi_\delta*\left(Q^2(z^{\varepsilon,\delta,\mu})BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu})\right)(\alpha,t)+\phi_\mu*\left(c^{\varepsilon,\delta,\mu}\left(\phi_\mu*\partial_\alpha z^{\varepsilon,\delta,\mu}\right)\right)(\alpha,t), \tag{2.102}$$

$$\begin{aligned}\omega_t^{\varepsilon,\delta,\mu}=&\frac{|\partial_\alpha z^{\varepsilon,\delta,\mu}|}{Q^2(z^{\varepsilon,\delta,\mu})}\phi_\delta*\phi_\delta*\left(\frac{Q^2(z^{\varepsilon,\delta,\mu})}{|\partial_\alpha z^{\varepsilon,\delta,\mu}(\alpha,t)|}\left[-2BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu})_t\cdot z_\alpha^{\varepsilon,\delta,\mu}\right.\right.\\&-2Q(z^{\varepsilon,\delta,\mu})Q(z^{\varepsilon,\delta,\mu})_\alpha|BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu})|^2-\partial_\alpha\left(\frac{(\varphi^{\varepsilon,\delta,\mu})^2}{Q(z^{\varepsilon,\delta,\mu})^2}\right)\\&+\frac{c^{\varepsilon,\delta,\mu}|z_\alpha^{\varepsilon,\delta,\mu}|^2}{\pi Q(z^{\varepsilon,\delta,\mu})^2}\int_{-\pi}^{\pi}(Q(z^{\varepsilon,\delta,\mu})^2BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu}))_\beta\cdot\frac{z_\beta^{\varepsilon,\delta,\mu}}{|z_\beta^{\varepsilon,\delta,\mu}|^2}d\beta\\&-\frac{4c^{\varepsilon,\delta,\mu}Q(z^{\varepsilon,\delta,\mu})_\alpha BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu})\cdot z_\alpha^{\varepsilon,\delta,\mu}}{Q(z^{\varepsilon,\delta,\mu})}\\&\left.\left.-\frac{2(c^{\varepsilon,\delta,\mu})^2|z_\alpha^{\varepsilon,\delta,\mu}|^2Q(z^{\varepsilon,\delta,\mu})_\alpha}{Q(z^{\varepsilon,\delta,\mu})^3}-2\partial_\alpha\left(gP_2^{-1}(z^{\varepsilon,\delta,\mu})\right)\right]\right)\\&-\frac{2|\partial_\alpha z^{\varepsilon,\delta,\mu}(\alpha,t)|}{Q^2(z^{\varepsilon,\delta,\mu})}\left(Q(z^{\varepsilon,\delta,\mu})\partial_tQ(z^{\varepsilon,\delta,\mu})\frac{\omega^{\varepsilon,\delta,\mu}}{|\partial_\alpha z^{\varepsilon,\delta,\mu}|}-\frac{Q^2(z^{\varepsilon,\delta,\mu})\omega^{\varepsilon,\delta,\mu}}{2|\partial_\alpha z^{\varepsilon,\delta,\mu}|^3}\partial_\alpha z^{\varepsilon,\delta,\mu}\cdot\partial_\alpha\partial_tz^{\varepsilon,\delta,\mu}\right)\\&+\frac{2|\partial_\alpha z^{\varepsilon,\delta,\mu}(\alpha,t)|}{Q^2(z^{\varepsilon,\delta,\mu})}\phi_\delta*\phi_\delta*\left(Q(z^{\varepsilon,\delta,\mu})\partial_tQ(z^{\varepsilon,\delta,\mu})\frac{\omega^{\varepsilon,\delta,\mu}}{|\partial_\alpha z^{\varepsilon,\delta,\mu}|}-\frac{Q^2(z^{\varepsilon,\delta,\mu})\omega^{\varepsilon,\delta,\mu}}{2|\partial_\alpha z^{\varepsilon,\delta,\mu}|^3}\partial_\alpha z^{\varepsilon,\delta,\mu}\cdot\partial_\alpha\partial_tz^{\varepsilon,\delta,\mu}\right)\\&-2\varepsilon\frac{|\partial_\alpha z^{\varepsilon,\delta,\mu}|}{Q^2(z^{\varepsilon,\delta,\mu})}\Lambda(\phi_\mu*\phi_\mu*\varphi^{\varepsilon,\delta,\mu}),\end{aligned} \tag{2.103}$$

$z^{\varepsilon,\delta,\mu}(\alpha,0)=z_0(\alpha)$ and $\omega^{\varepsilon,\delta,\mu}(\alpha,0)=\omega_0(\alpha)$ for $\varepsilon>0,\delta>0,\mu>0$, ϕ_δ and ϕ_μ even mollifiers, and

$$\begin{aligned}c^{\varepsilon,\delta,\mu}(\alpha)=&\frac{\alpha+\pi}{2\pi}\int_{-\pi}^{\pi}\frac{\partial_\beta z^{\varepsilon,\delta,\mu}(\beta))}{|\partial_\beta z^{\varepsilon,\delta,\mu}(\beta)|^2}\cdot\phi_\delta*\phi_\delta*(\partial_\beta(Q^2(z^{\varepsilon,\delta,\mu})(\beta)BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu}))(\beta))d\beta\\&-\int_{-\pi}^{\alpha}\frac{\partial_\beta z^{\varepsilon,\delta,\mu}(\beta)}{|\partial_\beta z^{\varepsilon,\delta,\mu}(\beta)|^2}\cdot\phi_\delta*\phi_\delta*(\partial_\beta(Q^2(z^{\varepsilon,\delta,\mu})(\beta)BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu}))(\beta))d\beta,\end{aligned}$$

$$\varphi^{\varepsilon,\delta,\mu}=\frac{Q^2(z^{\varepsilon,\delta,\mu})\omega^{\varepsilon,\delta,\mu}}{2|\partial_\alpha z^{\varepsilon,\delta,\mu}|}-\mathcal{C}^{\varepsilon,\delta,\mu},$$

$$B^{\varepsilon,\delta,\mu}(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial_\alpha z^{\varepsilon,\delta,\mu}(\alpha,t)}{|\partial_\alpha z^{\varepsilon,\delta,\mu}(\alpha,t)|^2}\cdot\partial_\alpha(Q^2(z^{\varepsilon,\delta,\mu})BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu}))(\alpha,t)d\alpha,$$

$$\begin{aligned}\mathcal{C}^{\varepsilon,\delta,\mu}=&\phi_\delta*\phi_\delta*\left(\frac{\alpha+\pi}{2\pi}\int_{-\pi}^{\pi}\frac{\partial_\beta z^{\varepsilon,\delta,\mu}(\beta)}{|\partial_\beta z^{\varepsilon,\delta,\mu}(\beta)|}\cdot(\partial_\beta(Q^2(z^{\varepsilon,\delta,\mu})BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu})))(\beta)d\beta\right)\\&-\phi_\delta*\phi_\delta*\left(\int_{-\pi}^{\alpha}\frac{\partial_\beta z^{\varepsilon,\delta,\mu}(\beta)}{|\partial_\beta z^{\varepsilon,\delta,\mu}(\beta)|}\cdot(\partial_\beta(Q^2(z^{\varepsilon,\delta,\mu})(\beta)BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu}))(\beta))d\beta\right).\end{aligned}$$

We start proving the following lemma:

Lemma 2.4.17 *Let $z^{\varepsilon,\delta,\mu}(\alpha,t) \in H^4(\mathbb{T})$, $\omega^{\varepsilon,\delta,\mu}(\alpha,t) \in H^2(\mathbb{T})$, $\varphi^{\varepsilon,\delta,\mu}(\alpha,t) \in H^3(\mathbb{T})$. Then $\omega^{\varepsilon,\delta,\mu}(\alpha,t) \in H^3(\mathbb{T})$.*

Proof: We can write $\omega^{\varepsilon,\delta,\mu}$ as:

$$\omega^{\varepsilon,\delta,\mu} = \frac{2|\partial_\alpha z^{\varepsilon,\delta,\mu}|}{Q^2(z^{\varepsilon,\delta,\mu})}\left(\varphi^{\varepsilon,\delta,\mu} + \mathcal{C}^{\varepsilon,\delta,\mu}\right).$$

Taking three derivatives yields

$$\begin{aligned}
\partial_\alpha^3\omega^{\varepsilon,\delta,\mu} &= \text{ SAFE } + \frac{2|\partial_\alpha z^{\varepsilon,\delta,\mu}|}{Q^2(z^{\varepsilon,\delta,\mu})}\partial_\alpha^3\mathcal{C}^{\varepsilon,\delta,\mu}\\
&= \text{ SAFE } - \frac{2|\partial_\alpha z^{\varepsilon,\delta,\mu}|}{Q^2(z^{\varepsilon,\delta,\mu})}\phi_\delta * \phi_\delta * \partial_\alpha^2\left(\frac{\partial_\alpha z^{\varepsilon,\delta,\mu}}{|\partial_\alpha z^{\varepsilon,\delta,\mu}|}\cdot(\partial_\alpha(Q^2(z^{\varepsilon,\delta,\mu})BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu})))\right)\\
&= \text{ SAFE } - \frac{2|\partial_\alpha z^{\varepsilon,\delta,\mu}|}{Q^2(z^{\varepsilon,\delta,\mu})}\phi_\delta * \phi_\delta * \left(\frac{\partial_\alpha z^{\varepsilon,\delta,\mu}}{|\partial_\alpha z^{\varepsilon,\delta,\mu}|}\cdot(Q^2(z^{\varepsilon,\delta,\mu})\partial_\alpha^3 BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu}))\right)\\
&= \text{ SAFE } - \frac{2|\partial_\alpha z^{\varepsilon,\delta,\mu}|}{Q^2(z^{\varepsilon,\delta,\mu})}\phi_\delta * \phi_\delta * \left(\frac{\partial_\alpha z^{\varepsilon,\delta,\mu}}{|\partial_\alpha z^{\varepsilon,\delta,\mu}|}\cdot(Q^2(z^{\varepsilon,\delta,\mu})BR(z^{\varepsilon,\delta,\mu},\partial_\alpha^3\omega^{\varepsilon,\delta,\mu}))\right)
\end{aligned}$$

where SAFE means bounded in L^2. Using the representation

$$BR(z^{\varepsilon,\delta,\mu},\partial_\alpha^3\omega^{\varepsilon,\delta,\mu}) = \text{ SAFE } + \frac{1}{2}\frac{(\partial_\alpha z^{\varepsilon,\delta,\mu})^\perp}{|\partial_\alpha z^{\varepsilon,\delta,\mu}|^2}H(\partial_\alpha^3\omega^{\varepsilon,\delta,\mu})$$

we get that

$$\partial_\alpha^3\omega^{\varepsilon,\delta,\mu} = \text{ SAFE } - \frac{2|\partial_\alpha z^{\varepsilon,\delta,\mu}|}{Q^2(z^{\varepsilon,\delta,\mu})}\phi_\delta * \phi_\delta * \underbrace{\left(Q^2(z^{\varepsilon,\delta,\mu})\frac{1}{2}\frac{(\partial_\alpha z^{\varepsilon,\delta,\mu})^\perp}{|\partial_\alpha z^{\varepsilon,\delta,\mu}|^2}H(\partial_\alpha^3\omega^{\varepsilon,\delta,\mu})\cdot\frac{\partial_\alpha z^{\varepsilon,\delta,\mu}}{|\partial_\alpha z^{\varepsilon,\delta,\mu}|}\right)}_{=0}$$

and we are done. We should remark that the lemma holds independently of δ, μ and ε. □

We define a distance between data (z,ω) and $(\underline{z},\underline{\omega})$ by taking

$$d((z,\omega),(\underline{z},\underline{\omega})) = \|z-\underline{z}\|_{H^4} + \|\omega-\underline{\omega}\|_{H^2} + \|\varphi-\underline{\varphi}\|_{H^3}$$

where φ and $\underline{\varphi}$ arise from (z,ω) and $(\underline{z},\underline{\omega})$ respectively by (2.62). Let XX denote the resulting metric space. The proof of Lemma 2.4.17 gives also the following

Corollary 2.4.18 *The map $(z,\omega)\mapsto\omega$ is Lipschitz from any ball in XX into $H^3(\mathbb{T})$.*

We note that throughout this section we will repeatedly use the following commutator estimate for convolutions:

$$\|\phi_\delta * (\partial_\alpha f g) - g\phi_\delta * (\partial_\alpha f)\|_{L^2} \le C\|\partial_\alpha g\|_{L^\infty}\|f\|_{L^2}, \tag{2.104}$$

where the constant C is independent of δ, f and g. We can now operate to get the following expression for $\varphi^{\varepsilon,\delta,\mu}$:

$$\begin{aligned}
\partial_t\varphi^{\varepsilon,\delta,\mu} &= \frac{Q(z^{\varepsilon,\delta,\mu})\partial_t Q(z^{\varepsilon,\delta,\mu})\omega^{\varepsilon,\delta,\mu}}{|\partial_\alpha z^{\varepsilon,\delta,\mu}|} - \frac{Q^2(z^{\varepsilon,\delta,\mu})\omega^{\varepsilon,\delta,\mu}}{2|\partial_\alpha z^{\varepsilon,\delta,\mu}|^3}\partial_\alpha z^{\varepsilon,\delta,\mu}\cdot\partial_t\partial_\alpha z^{\varepsilon,\delta,\mu} + \frac{Q^2(z^{\varepsilon,\delta,\mu})\partial_t\omega^{\varepsilon,\delta,\mu}}{2|\partial_\alpha z^{\varepsilon,\delta,\mu}|} - \partial_t\mathcal{C}^{\varepsilon,\delta,\mu}\\
&= \phi_\delta*\phi_\delta*\left(\frac{Q^2(z^{\varepsilon,\delta,\mu})}{2|\partial_\alpha z^{\varepsilon,\delta,\mu}(\alpha,t)|}\left[-2BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu})_t\cdot z_\alpha^{\varepsilon,\delta,\mu}\right.\right.\\
&- 2Q(z^{\varepsilon,\delta,\mu})Q(z^{\varepsilon,\delta,\mu})_\alpha|BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu})|^2 - \partial_\alpha\left(\frac{(\varphi^{\varepsilon,\delta,\mu})^2}{Q(z^{\varepsilon,\delta,\mu})^2}\right)\\
&+ \frac{c^{\varepsilon,\delta,\mu}|z_\alpha^{\varepsilon,\delta,\mu}|^2}{\pi Q(z^{\varepsilon,\delta,\mu})^2}\int_{-\pi}^{\pi}(Q(z^{\varepsilon,\delta,\mu})^2 BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu}))_\beta\cdot\frac{z_\beta^{\varepsilon,\delta,\mu}}{|z_\beta^{\varepsilon,\delta,\mu}|^2}d\beta\\
&- \frac{4c^{\varepsilon,\delta,\mu}Q(z^{\varepsilon,\delta,\mu})_\alpha BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu})\cdot z_\alpha^{\varepsilon,\delta,\mu}}{Q(z^{\varepsilon,\delta,\mu})}\\
&\left.\left.-\frac{2(c^{\varepsilon,\delta,\mu})^2|z_\alpha^{\varepsilon,\delta,\mu}|^2 Q(z^{\varepsilon,\delta,\mu})_\alpha}{Q(z^{\varepsilon,\delta,\mu})^3} - 2\partial_\alpha\left(gP_2^{-1}(z^{\varepsilon,\delta,\mu})\right)\right]\right)\\
&+ \phi_\delta*\phi_\delta*\left(Q(z^{\varepsilon,\delta,\mu})\partial_t Q(z^{\varepsilon,\delta,\mu})\frac{\omega^{\varepsilon,\delta,\mu}}{|\partial_\alpha z^{\varepsilon,\delta,\mu}|} - \frac{Q^2(z^{\varepsilon,\delta,\mu})\omega^{\varepsilon,\delta,\mu}}{2|\partial_\alpha z^{\varepsilon,\delta,\mu}|^3}\partial_\alpha z^{\varepsilon,\delta,\mu}\cdot\partial_\alpha\partial_t z^{\varepsilon,\delta,\mu}\right)\\
&- \varepsilon\Lambda(\phi_\mu*\phi_\mu*\varphi^{\varepsilon,\delta,\mu}) - \partial_t\mathcal{C}^{\varepsilon,\delta,\mu}.
\end{aligned}$$

The RHS of the evolution equations for $z^{\varepsilon,\delta,\mu}$ and $\varphi^{\varepsilon,\delta,\mu}$ are Lipschitz in the spaces $H^4(\mathbb{T})$ and $H^{3+\frac{1}{2}}(\mathbb{T})$ since they are mollified. For the case of $\omega^{\varepsilon,\delta,\mu}$ (Lipschitz in the space $H^2(\mathbb{T})$) we use that for δ small enough $\phi_\delta*\phi_\delta$ is close to the identity and the a priori bounds. In all of the cases we have taken advantage of Lemma 2.4.17. Therefore we can solve (3.9-3.10) for short time, thanks to Picard's theorem.

Now, we can perform energy estimates as in the a priori case to get uniform bounds in μ and we can let μ go to zero. The energy estimates that we can get are the following:

$$\begin{aligned}
&\frac{d}{dt}\left(\|z^{\varepsilon,\delta,\mu}\|_{H^4}^2 + \|\mathcal{F}(z^{\varepsilon,\delta,\mu})\|_{L^\infty}^2 + \|\omega^{\varepsilon,\delta,\mu}\|_{H^2}^2 + \|\varphi^{\varepsilon,\delta,\mu}\|_{H^{3+\frac{1}{2}}}^2 + \sum_{l=0}^4\frac{1}{m^{\varepsilon,\delta,\mu}(q^l)}\right)(t)\\
&\leq C(\varepsilon,\delta)\left(\|z^{\varepsilon,\delta,\mu}\|_{H^4}^2 + \|\mathcal{F}(z^{\varepsilon,\delta,\mu})\|_{L^\infty}^2 + \|\omega^{\varepsilon,\delta,\mu}\|_{H^2}^2 + \|\varphi^{\varepsilon,\delta,\mu}\|_{H^{3+\frac{1}{2}}}^2 + \sum_{l=0}^4\frac{1}{m^{\varepsilon,\delta,\mu}(q^l)}\right)^j(t).
\end{aligned}$$

We should note that for the new system without the ϕ_μ mollifier, the length of the tangent vector $|\partial_\alpha z^{\varepsilon,\delta}|$ is now constant in space and depends only on time. Lemma 2.4.17 still applies and we can still perform energy estimates as in the a priori case. The only difference relies on the fact that we should have to move the mollifiers and apply the estimate (3.11). We should also remark that because of the dissipative term $\varepsilon\Lambda\varphi^{\varepsilon,\delta}$ it is enough to use the following estimate

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{1/2}(\partial_\alpha^3\varphi^{\varepsilon,\delta})\|_{L^2}^2 = \int \Lambda(\partial_\alpha^3\varphi^{\varepsilon,\delta})\partial_t\partial_\alpha^3\varphi^{\varepsilon,\delta}d\alpha \leq \frac{\varepsilon}{64}\|\Lambda(\partial_\alpha^3\varphi^{\varepsilon,\delta})\|_{L^2}^2 + C(\varepsilon)\|\partial_t\partial_\alpha^3\varphi^{\varepsilon,\delta}\|_{L^2}^2$$

and hence require only that $\partial_t\varphi^{\varepsilon,\delta} \in H^3(\mathbb{T})$ (instead of the $H^{3+\frac{1}{2}}(\mathbb{T})$ that was required before) except for the transport term that can be estimated as in subsection 2.4.4.12. The estimations are performed following exactly the same steps of subsection 2.4.4. More precisely, we can get the following energy estimates:

$$\begin{aligned}
&\frac{d}{dt}\left(\|z^{\varepsilon,\delta}\|_{H^4}^2 + \|\mathcal{F}(z^{\varepsilon,\delta})\|_{L^\infty}^2 + \|\omega^{\varepsilon,\delta}\|_{H^2}^2 + \|\varphi^{\varepsilon,\delta}\|_{H^{3+\frac{1}{2}}}^2 + \sum_{l=0}^{4}\frac{1}{m^{\varepsilon,\delta}(q^l)}\right)(t)\\
&\leq C(\varepsilon)\left(\|z^{\varepsilon,\delta}\|_{H^4}^2 + \|\mathcal{F}(z^{\varepsilon,\delta})\|_{L^\infty}^2 + \|\omega^{\varepsilon,\delta}\|_{H^2}^2 + \|\varphi^{\varepsilon,\delta}\|_{H^{3+\frac{1}{2}}}^2 + \sum_{l=0}^{4}\frac{1}{m^{\varepsilon,\delta}(q^l)}\right)^j(t).
\end{aligned}$$

Under these conditions, we can let δ go to zero.

Finally, let $z^\varepsilon(\alpha,t)$ be a solution of the following system (compare with (2.64)):

$$z_t^\varepsilon(\alpha,t) = Q^2(z^\varepsilon)(\alpha,t)BR(z^\varepsilon,\omega^\varepsilon)(\alpha,t) + c^\varepsilon(\alpha,t)\partial_\alpha z^\varepsilon(\alpha,t), \tag{2.105}$$

$$\begin{aligned}
\omega_t^\varepsilon =& -2BR(z^\varepsilon,\omega^\varepsilon)_t\cdot z_\alpha^\varepsilon - 2Q(z^\varepsilon)Q(z^\varepsilon)_\alpha|BR(z^\varepsilon,\omega^\varepsilon)|^2 - \partial_\alpha\left(\frac{(\varphi^\varepsilon)^2}{Q(z^\varepsilon)^2}\right)\\
&+\frac{c^\varepsilon|z_\alpha^\varepsilon|^2}{\pi Q(z^\varepsilon)^2}\int_{-\pi}^{\pi}(Q(z^\varepsilon)^2BR(z^\varepsilon,\omega^\varepsilon))_\beta\cdot\frac{z_\beta^\varepsilon}{|z_\beta^\varepsilon|^2}d\beta - \frac{4c^\varepsilon Q(z^\varepsilon)_\alpha BR(z^\varepsilon,\omega^\varepsilon)\cdot z_\alpha^\varepsilon}{Q(z^\varepsilon)}\\
&-\frac{2(c^\varepsilon)^2|z_\alpha^\varepsilon|^2Q(z^\varepsilon)_\alpha}{Q(z^\varepsilon)^3} - 2\partial_\alpha\left(gP_2^{-1}(z^\varepsilon)\right) - 2\varepsilon\frac{|\partial_\alpha z^\varepsilon|}{Q^2(z^\varepsilon)}\Lambda\varphi^\varepsilon,
\end{aligned}\tag{2.106}$$

$z^\varepsilon(\alpha,0) = z_0(\alpha)$ and $\omega^\varepsilon(\alpha,0) = \omega_0(\alpha)$ for $\varepsilon > 0$, where

$$\begin{aligned}
c^\varepsilon(\alpha) =& \frac{\alpha+\pi}{2\pi}\int_{-\pi}^{\pi}\frac{\partial_\beta z^\varepsilon(\beta))}{|\partial_\beta z^\varepsilon(\beta)|^2}\cdot\partial_\beta(Q^2(z^\varepsilon)(\beta)BR(z^\varepsilon,\omega^\varepsilon)(\beta))d\beta\\
&-\int_{-\pi}^{\alpha}\frac{\partial_\beta z^\varepsilon(\beta)}{|\partial_\beta z^\varepsilon(\beta)|^2}\cdot\partial_\beta(Q^2(z^\varepsilon)(\beta)BR(z^\varepsilon,\omega^\varepsilon)(\beta))d\beta,
\end{aligned}$$

$$\varphi^\varepsilon = \frac{Q^2(z^\varepsilon)\omega^\varepsilon}{2|\partial_\alpha z^\varepsilon|} - |\partial_\alpha z^\varepsilon|c^\varepsilon, \qquad B^\varepsilon(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial_\alpha z^\varepsilon(\alpha,t)}{|\partial_\alpha z^\varepsilon(\alpha,t)|^2}\cdot\partial_\alpha(Q^2(z^\varepsilon)BR(z^\varepsilon,\omega^\varepsilon))(\alpha,t)d\alpha.$$

Proceeding as in section 2.4.3 (compare with equation (2.87)) we find

$$
\begin{aligned}
\partial_\alpha \varphi_t^\varepsilon =& - B^\varepsilon(t)\varphi_\alpha^\varepsilon - \frac{\partial_\alpha^2((\varphi^\varepsilon)^2)}{2|\partial_\alpha z^\varepsilon|} + \partial_\alpha\left(\frac{\partial_\alpha Q(z^\varepsilon)}{|\partial_\alpha z^\varepsilon| Q(z^\varepsilon)}(\varphi^\varepsilon)^2\right)\\
&- Q(z^\varepsilon)^2\partial_t BR(z^\varepsilon,\omega^\varepsilon)\cdot\partial_\alpha^\perp z^\varepsilon \frac{\partial_\alpha^2 z^\varepsilon\cdot\partial_\alpha^\perp z^\varepsilon}{|\partial_\alpha z^\varepsilon|^3} - \partial_t(|\partial_\alpha z^\varepsilon| B^\varepsilon(t))\\
&+ \partial_\alpha(Q(z^\varepsilon)^2 BR(z^\varepsilon,\omega^\varepsilon))\cdot\partial_\alpha^\perp z^\varepsilon \frac{\partial_t\partial_\alpha z^\varepsilon\cdot\partial_\alpha^\perp z^\varepsilon}{|\partial_\alpha z^\varepsilon|^3} + 2\partial_\alpha(Q(z^\varepsilon)\partial_t Q(z^\varepsilon)BR(z^\varepsilon,\omega^\varepsilon))\cdot\frac{\partial_\alpha z^\varepsilon}{|\partial_\alpha z^\varepsilon|}\\
&- \partial_\alpha\left(Q(z^\varepsilon)^2\frac{\partial_\alpha(gP_2^{-1}(z^\varepsilon))}{|\partial_\alpha z^\varepsilon|}\right) + \partial_\alpha\left(Q(z^\varepsilon)\partial_t Q(z^\varepsilon)\frac{\omega^\varepsilon}{|\partial_\alpha z^\varepsilon|}\right)\\
&- \partial_\alpha\left(2c^\varepsilon BR(z^\varepsilon,\omega^\varepsilon)\cdot\frac{\partial_\alpha z^\varepsilon}{|\partial_\alpha z^\varepsilon|}Q(z^\varepsilon)\partial_\alpha Q(z^\varepsilon)\right) - \partial_\alpha\left(\frac{\partial_\alpha Q(z^\varepsilon)}{Q(z^\varepsilon)}(c^\varepsilon)^2|\partial_\alpha z^\varepsilon|\right)\\
&- \partial_\alpha\left(\frac{Q(z^\varepsilon)^3}{|\partial_\alpha z^\varepsilon|}|BR(z^\varepsilon,\omega^\varepsilon)|^2\partial_\alpha Q(z^\varepsilon)\right) - \varepsilon\Lambda\partial_\alpha\varphi^\varepsilon.
\end{aligned}
\tag{2.107}
$$

We also define (compare with equation (5.2))

$$
\begin{aligned}
\sigma^\varepsilon =& (\partial_t BR(z^\varepsilon,\omega^\varepsilon) + \frac{\varphi^\varepsilon}{|\partial_\alpha z^\varepsilon|}\partial_\alpha BR(z^\varepsilon,\omega^\varepsilon))\cdot\partial_\alpha^\perp z^\varepsilon + \frac{1}{2}\frac{\omega^\varepsilon}{|\partial_\alpha z^\varepsilon|^2}(\partial_\alpha z_t^\varepsilon + \frac{\varphi^\varepsilon}{|\partial_\alpha z^\varepsilon|}\partial_\alpha^2 z^\varepsilon)\cdot\partial_\alpha^\perp z^\varepsilon\\
&+ Q(z^\varepsilon)\left|BR(z^\varepsilon,\omega^\varepsilon) + \frac{\omega^\varepsilon}{2|\partial_\alpha z^\varepsilon|^2}\partial_\alpha z^\varepsilon\right|^2(\nabla Q(z^\varepsilon))\cdot\partial_\alpha^\perp z^\varepsilon + g(\nabla P_2^{-1}(z^\varepsilon))\cdot\partial_\alpha^\perp z^\varepsilon.
\end{aligned}
$$

Remark 2.4.19 *The system (2.105-2.106) is analogous to the system considered in [28, Section 8]. We point out an unfortunate typographical error in that section; the Laplacian should have been written as the square root of the Laplacian.*

For this ε-system (2.105-2.106) we now know that there is local-existence for initial data satisfying $\mathcal{F}(z_0)(\alpha,\beta) < \infty$ even if $\sigma^\varepsilon(\alpha,0)$ does not have the proper sign. In the following we shall show briefly how to obtain a solution of the regularized system with $z^\varepsilon \in C([0,T^\varepsilon],H^k), \varphi^\varepsilon \in C([0,T^\varepsilon],H^{k-\frac{1}{2}}), \omega^\varepsilon \in C([0,T^\varepsilon],H^{k-2})$ for $k \geq 4$.

The next step is to integrate the system during a time T independent of ε. We will show that for this system we have

$$
\frac{d}{dt}E(t) \leq CE^p(t), \tag{2.108}
$$

where $E(t)$ is given by the analogous formula (2.67) for the ε-system, and C and p are constants independent of ε.

In the following we shall see what is the impact of the ε system on the a priori estimates and check that there is no practical impact for sufficiently small ε. To do that, we will show the corresponding uniform estimates for $k = 4$ and leave to the reader the remaining easier cases. Let us consider the one corresponding to I_3 in section 2.4.4.12, we have

$$
I_3^\varepsilon = -2\int_{-\pi}^{\pi}\frac{1}{|\partial_\alpha z^\varepsilon|^3}\Lambda(\partial_\alpha^3\varphi^\varepsilon(\alpha))\partial_\alpha^2(Q^2(z^\varepsilon)\sigma^\varepsilon\partial_\alpha^2 z^\varepsilon\cdot\partial_\alpha^\perp z^\varepsilon)d\alpha.
$$

Proceeding in the same way as before, we can perform the same splittings and get uniform bounds such that $I_3^\varepsilon = -S^\varepsilon + M_4^\varepsilon +$ "bounded terms" where S^ε corresponds to S in (2.79),

$$|\text{“bounded terms”}| \le CE^p(t),$$

and

$$M_4^\varepsilon = -2\varepsilon \int_{-\pi}^{\pi} \frac{\partial_\alpha^2 z^\varepsilon \cdot \partial_\alpha^\perp z^\varepsilon}{|\partial_\alpha z^\varepsilon|^2} H(\partial_\alpha^3 \varphi^\varepsilon) H(\Lambda \partial_\alpha^3 \varphi^\varepsilon) d\alpha.$$

Then we can write M_4^ε as follows

$$M_4^\varepsilon = -2\varepsilon \int_{-\pi}^{\pi} \Lambda^{\frac{1}{2}} \Big(\frac{\partial_\alpha^2 z^\varepsilon \cdot \partial_\alpha^\perp z^\varepsilon}{|\partial_\alpha z^\varepsilon|^2} H(\partial_\alpha^3 \varphi^\varepsilon) \Big) \Lambda^{\frac{1}{2}} (H \partial_\alpha^3 \varphi^\varepsilon) d\alpha,$$

and therefore, for small ε

$$M_4^\varepsilon \le \|\Lambda^{\frac{1}{2}} \partial_\alpha^3 \varphi^\varepsilon\|_{L^2}^2 + \text{“bounded terms”},$$

which gives

$$\frac{d}{dt} E(t) \le CE^p(t) - \frac{\varepsilon}{2} \|\Lambda(\partial_\alpha^3 \varphi)\|_{L^2}^2.$$

This finally shows (2.108) and therefore

$$E(t) \le (Ct(1-p) + E^{1-p}(0))^{1/(1-p)}.$$

Now we are in position to extend the time of existence T^ε so long as the above estimate works and obtain a time T dependent only on the initial data (arc-chord, Rayleigh-Taylor, distance to the points $q^0, \ldots, q^4$, and Sobolev norms of z, ω, and φ). We can let ε tend to 0, and get a solution of the original system. This concludes the proof.

Chapter 3

Splash singularities for water waves with surface tension

3.1 Introduction

We establish the main result in the chapter for the system (1.1-1.4) and (1.6).

Theorem 3.1.1 *Consider $z_0(\alpha) - (\alpha, 0) \in H^k(\mathbb{T})$ for $k \geq 5$. Then there exist a family of initial data satisfying* (1.5) *and the arc-chord condition* (1.7) *and a time $T_s > 0$ such that the interface $z(\alpha, t) \in H^k(\mathbb{T})$ from the unique smooth solution of the system (1.1-1.7) on the time interval $[0, T_s]$ touches itself at a single point ("splash" singularity) or along an arc ("splat" singularity) at time $t = T_s$.*

These solutions can be extended to the periodic $3D$ setting considering scenarios invariant under translations in one coordinate direction.

The strategy of the proof of the main result is to establish a local existence theorem from the initial data that has a splash or a splat singularity (notice that the equations are time reversible invariant). Since the curve self-intersects (failure of the arc-chord condition), it is not clear if the amplitude of the vorticity remains smooth and the meaning of equations (3.1-3.2). In order to deal with these obstacles we use a conformal map

$$P(w) = \Big(\tan\Big(\frac{w}{2}\Big)\Big)^{1/2}, \quad w \in \mathbb{C},$$

whose intention is to keep apart the self-intersecting points taking the branch of the square root above passing through those crucial points. Here $P(z)$ will refer to a 2 dimensional vector whose components are the real and imaginary parts of $P(z_1 + iz_2)$. We also make sure that $\Omega(t) \cup \partial\Omega(t)$ do not contain any singular point of the transformation P. Then potential theory helps us to get the following analogous evolution equations for the new curve

$$\tilde{z}(\alpha, t) = P(z(\alpha, t))$$

and the new amplitude $\tilde{\omega}$:

$$\tilde{z}_t(\alpha, t) = Q^2(\alpha, t) BR(\tilde{z}, \tilde{\omega})(\alpha, t) + \tilde{c}(\alpha, t)\tilde{z}_\alpha(\alpha, t), \tag{3.1}$$

$$
\begin{aligned}
\tilde{\omega}_t(\alpha,t) =& -2BR_t(\tilde{z},\tilde{\omega})(\alpha,t)\cdot\tilde{z}_\alpha(\alpha,t) - |BR(\tilde{z},\tilde{\omega})|^2(Q^2)_\alpha(\alpha,t) - \Big(\frac{Q^2(\alpha,t)\tilde{\omega}(\alpha,t)^2}{4|\tilde{z}_\alpha(\alpha,t)|^2}\Big)_\alpha \\
&+ 2\tilde{c}(\alpha,t)BR_\alpha(\tilde{z},\tilde{\omega})\cdot\tilde{z}_\alpha(\alpha,t) + (\tilde{c}(\alpha,t)\tilde{\omega}(\alpha,t))_\alpha - 2\left(P_2^{-1}(\tilde{z}(\alpha,t))\right)_\alpha \\
&+ \tau\left(\frac{Q^3}{|\tilde{z}_\alpha(\alpha,t)|^3}(\tilde{z}_\alpha^T HP_2^{-1}\tilde{z}_\alpha\nabla P_1^{-1}\cdot\tilde{z}_\alpha - \tilde{z}_\alpha^T HP_1^{-1}\tilde{z}_\alpha\nabla P_2^{-1}\cdot\tilde{z}_\alpha)\right)_\alpha \\
&+ \tau\left(Q\frac{\tilde{z}_{\alpha\alpha}(\alpha,t)\cdot\tilde{z}_\alpha^\perp(\alpha,t)}{|\tilde{z}_\alpha(\alpha,t)|^3}\right)_\alpha
\end{aligned} \tag{3.2}
$$

where

$$
Q^2(\alpha,t) = \left|\frac{dP}{dw}(P^{-1}(\tilde{z}(\alpha,t)))\right|^2,
$$

and HP_i^{-1} denotes the Hessian matrix of P_i^{-1}, which is the i-th ($i=\{1,2\}$) component of the transformation P^{-1}.

Here, we choose $\tilde{c}(\alpha,t)$ in such a way that $|\tilde{z}_\alpha(\alpha,t)| = A(t)$. This particular choice of $\tilde{c}$ was first introduced by Hou et al. in [62] and was later used by Ambrose [6] and Ambrose-Masmoudi [8]. The choice of $\tilde{c}$ implies

$$
\begin{aligned}
\tilde{c}(\alpha,t) = &\frac{\alpha+\pi}{2\pi}\int_{-\pi}^{\pi}(Q^2BR(\tilde{z},\tilde{\omega}))_\beta(\beta,t)\cdot\frac{\tilde{z}_\beta(\beta,t)}{|\tilde{z}_\beta(\beta,t)|^2}d\beta \\
&- \int_{-\pi}^{\alpha}(Q^2BR(\tilde{z},\tilde{\omega}))_\beta(\beta,t)\cdot\frac{\tilde{z}_\beta(\beta,t)}{|\tilde{z}_\beta(\beta,t)|^2}d\beta
\end{aligned}
$$

It is easy to check that if we take $Q\equiv 1$ in (3.1-3.2) we recover (3.1-3.2).

We also define the function

$$
\tilde{\varphi}(\alpha,t) = \frac{Q^2(\alpha,t)\tilde{\omega}(\alpha,t)}{2|\tilde{z}_\alpha(\alpha,t)|} - \tilde{c}(\alpha,t)|\tilde{z}_\alpha(\alpha,t)| \tag{3.3}
$$

introduced by Beale et al. for the linear case [14] and by Ambrose-Masmoudi for the nonlinear one [8]. This function will be used to prove local existence in Sobolev spaces.

In the sections below, we show a local existence theorem based on energy estimates. Section 3.3 is devoted to provide the appropriate initial data for the splash and splat singularities. In Section 3.4 we choose an energy which does not need a precise sign on the Rayleigh-Taylor function. In Section 3.5 we choose a different energy that involves the sign of the Rayleigh-Taylor function and the estimates are uniform with respect to the surface tension coefficient. These two energies are based on the ones obtained in the non-tilde domain by Ambrose ([6]) and Ambrose-Masmoudi ([9]).

The Rayleigh-Taylor function is given by the following formula

$$
\begin{aligned}
\sigma \equiv &\left(BR_t(\tilde{z},\tilde{\omega}) + \frac{\tilde{\varphi}}{|\tilde{z}_\alpha|}BR_\alpha(\tilde{z},\tilde{\omega})\right)\cdot\tilde{z}_\alpha^\perp + \frac{\tilde{\omega}}{2|\tilde{z}_\alpha|^2}\left(\tilde{z}_{\alpha t} + \frac{\tilde{\varphi}}{|\tilde{z}_\alpha|}\tilde{z}_{\alpha\alpha}\right)\cdot\tilde{z}_\alpha^\perp \\
&+ Q\left|BR(\tilde{z},\tilde{\omega}) + \frac{\tilde{\omega}}{2|\tilde{z}_\alpha|^2}\tilde{z}_\alpha\right|^2(\nabla Q)(\tilde{z})\cdot\tilde{z}_\alpha^\perp + (\nabla P_2^{-1})(\tilde{z})\cdot\tilde{z}_\alpha^\perp.
\end{aligned} \tag{3.4}
$$

All solutions that we will consider throughout the paper will have finite energy, as discussed in [20]. The system satisfies the conservation of the mechanical energy. We define it this way: (not to be confused with the subsequent definitions of some other energies, see sections 3.4 and 3.5).

$$
\begin{aligned}
\mathcal{E}_S(t) &= \frac{1}{2}\int_{\Omega_f(t)} |v(x,y,t)|^2 dxdy + \frac{1}{2}\int_{-\pi}^{\pi} (z_2(\alpha,t))^2 \partial_\alpha z_1(\alpha,t) d\alpha + \frac{\tau}{2}\int_{-\pi}^{\pi} |\partial_\alpha z(\alpha,t)| d\alpha \\
&\equiv \mathcal{E}_k(t) + \mathcal{E}_p(t) + \mathcal{E}_\tau(t),
\end{aligned}
$$

where $z(\alpha,t) = (z_1(\alpha,t), z_2(\alpha,t)), u(\alpha,t) = v(z(\alpha,t),t)$, and $\Omega_f(t) = \Omega(t) \cap [-\pi,\pi] \times \mathbb{R}$ is a fundamental domain in the water region in a period, then it follows that the energy is conserved.

$$
\begin{aligned}
\frac{d\mathcal{E}_k(t)}{dt} &= \int_{\Omega_f(t)} v(x,y,t)(v_t(x,y,t) + v(x,y,t)\cdot\nabla v(x,y,t))dxdy \\
&= \int_{\Omega_f(t)} v(x,y,t)(-\nabla p(x,y,t) - (0,1))dxdy \\
&= -\int_{\Omega_f(t)} v(x,y,t)(\nabla(p(x,y,t) + y))dxdy \\
&= -\int_{\partial(\Omega_f(t))} v(x,y,t)\cdot\overrightarrow{n} y ds + \int_{\partial(\Omega_f(t))} v(x,y,t)\cdot\overrightarrow{n}\frac{\tau}{2}K ds \\
&= -\int_{-\pi}^{\pi} z_2(\alpha,t)u(\alpha,t)\cdot\partial_\alpha z^\perp(\alpha,t)d\alpha + \frac{\tau}{2}\int_{-\pi}^{\pi} u(\alpha,t)\cdot\partial_\alpha z^\perp(\alpha,t)\frac{\partial_\alpha^2 z(\alpha,t)\cdot\partial_\alpha z^\perp(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^3}d\alpha
\end{aligned}
\tag{3.5}
$$

where we have used the incompressibility of the fluid ($\nabla\cdot v = 0$) and Laplace-Young's condition for the pressure on the interface. Next

$$
\begin{aligned}
\frac{d\mathcal{E}_p(t)}{dt} &= \int_{-\pi}^{\pi} z_2(\alpha,t)\partial_t z_2(\alpha,t)\partial_\alpha z_1(\alpha,t)d\alpha + \frac{1}{2}\int_{-\pi}^{\pi}(z_2(\alpha,t))^2\partial_t\partial_\alpha z_1(\alpha,t)d\alpha \\
&= \int_{-\pi}^{\pi} z_2(\alpha,t)\partial_t z_2(\alpha,t)\partial_\alpha z_1(\alpha,t)d\alpha - \int_{-\pi}^{\pi} z_2(\alpha,t)\partial_\alpha z_2(\alpha,t)\partial_t z_1(\alpha,t)d\alpha \\
&= \int_{-\pi}^{\pi} z_2(\alpha,t)u(\alpha,t)\cdot\partial_\alpha z^\perp(\alpha,t)d\alpha.
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
\frac{d\mathcal{E}_\tau(t)}{dt} &= \frac{\tau}{2}\int_{-\pi}^{\pi}\frac{\partial_\alpha z(\alpha,t)\cdot\partial_\alpha\partial_t z(\alpha,t)}{|\partial_\alpha z(\alpha,t)|}d\alpha = -\frac{\tau}{2}\int_{-\pi}^{\pi}\frac{\partial_\alpha^2 z(\alpha,t)\cdot\partial_t z(\alpha,t)}{|\partial_\alpha z(\alpha,t)|}d\alpha \\
&= -\frac{\tau}{2}\int_{-\pi}^{\pi}\frac{\partial_\alpha^2 z(\alpha,t)\cdot u(\alpha,t)}{|\partial_\alpha z(\alpha,t)|}d\alpha = -\frac{\tau}{2}\int_{-\pi}^{\pi}\frac{\partial_\alpha^2 z(\alpha,t)\cdot\partial_\alpha z^\perp(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^3}u(\alpha,t)\cdot\partial_\alpha^\perp z(\alpha,t)d\alpha
\end{aligned}
\tag{3.7}
$$

Adding all the derivatives we get the desired result.

3.2 Properties of the curvature in the tilde domain

In this section we will rewrite the term corresponding to the curvature $K(z(\alpha,t))$ in the new tilde variables $\tilde{z}(\alpha,t)$.

We will proceed step by step. Let us recall that the curvature is defined by

$$K(\alpha,t) = \frac{z_{\alpha\alpha}(\alpha,t)\cdot z_{\alpha}^{\perp}(\alpha,t)}{|z_{\alpha}(\alpha,t)|^3}$$

We begin with the term $|z_{\alpha}(\alpha,t)|^3$. We have that

$$|\tilde{z}_{\alpha}(\alpha,t)|^2 = \langle \partial_{\alpha}P(z(\alpha,t)), \partial_{\alpha}P(z(\alpha,t))\rangle = \langle \nabla P(z(\alpha,t))\cdot z_{\alpha}(\alpha,t), \nabla P(z(\alpha,t))\cdot z_{\alpha}(\alpha,t)\rangle$$

Since P and P^{-1} are conformal, by the Cauchy-Riemann equations

$$\nabla P(z(\alpha,t))^T\nabla P(z(\alpha,t)) = Q^2(\alpha,t)\mathrm{Id}_2,$$

that implies that

$$|\tilde{z}_{\alpha}(\alpha,t)|^3 = Q^3(\alpha,t)|z_{\alpha}(\alpha,t)|^3$$

We move to the other term

$$\begin{aligned}\langle z_{\alpha\alpha}(\alpha,t), z_{\alpha}^{\perp}(\alpha,t)\rangle &= \langle \partial_{\alpha}\left(\nabla P^{-1}(\tilde{z}(\alpha,t))\cdot\tilde{z}_{\alpha}(\alpha,t)\right), (\nabla P^{-1}(\tilde{z}(\alpha,t))\cdot\tilde{z}_{\alpha}(\alpha,t))^{\perp}\rangle \\ &= \langle \nabla P^{-1}(\tilde{z}(\alpha,t))\cdot\tilde{z}_{\alpha\alpha}(\alpha,t), (\nabla P^{-1}(\tilde{z}(\alpha,t))\cdot\tilde{z}_{\alpha}(\alpha,t))^{\perp}\rangle \\ &+ \langle \partial_{\alpha}\left(\nabla P^{-1}(\tilde{z}(\alpha,t))\right)\cdot\tilde{z}_{\alpha}(\alpha,t), (\nabla P^{-1}(\tilde{z}(\alpha,t))\cdot\tilde{z}_{\alpha}(\alpha,t))^{\perp}\rangle \equiv W + X\end{aligned}$$

Again, by the Cauchy-Riemann equations

$$W = \frac{1}{Q^2(\alpha,t)}\langle \tilde{z}_{\alpha\alpha}(\alpha,t), \tilde{z}_{\alpha}(\alpha,t)^{\perp}\rangle$$

Developing the terms in X, we get

$$\left(\nabla P^{-1}(\tilde{z}(\alpha,t))\right)\cdot\tilde{z}_{\alpha}(\alpha,t) = \begin{pmatrix} \tilde{z}_{\alpha}^T(\alpha,t)\cdot HP_1^{-1}(\tilde{z}(\alpha,t))\cdot\tilde{z}_{\alpha}(\alpha,t) \\ \tilde{z}_{\alpha}^T(\alpha,t)\cdot HP_2^{-1}(\tilde{z}(\alpha,t))\cdot\tilde{z}_{\alpha}(\alpha,t)\end{pmatrix},$$

where HP_i^{-1} denotes the Hessian of the i-th component of P^{-1} $(i = 1,2)$. Hence, we can write X as

$$\begin{aligned}X = &-\tilde{z}_{\alpha}^T(\alpha,t)\cdot HP_1^{-1}(\tilde{z}(\alpha,t))\cdot\tilde{z}_{\alpha}(\alpha,t)\nabla P_2^{-1}(\tilde{z}(\alpha,t))\cdot\tilde{z}(\alpha,t) \\ &+ \tilde{z}_{\alpha}^T(\alpha,t)\cdot HP_2^{-1}(\tilde{z}(\alpha,t))\cdot\tilde{z}_{\alpha}(\alpha,t)\nabla P_1^{-1}(\tilde{z}(\alpha,t))\cdot\tilde{z}(\alpha,t).\end{aligned}$$

This means that

$$K(\alpha,t) = Q(\alpha,t)\frac{\tilde{z}_{\alpha\alpha}(\alpha,t)\cdot\tilde{z}_{\alpha}^{\perp}(\alpha,t)}{|\tilde{z}(\alpha,t)|^3} + X(\alpha,t)\frac{Q(\alpha,t)^3}{|\tilde{z}(\alpha,t)|^3} \equiv Q(\alpha,t)\tilde{K}(\alpha,t) + M(\alpha,t)$$

We will now try to simplify further by exploiting the Cauchy-Riemann equations. We can calculate the Hessian and the gradient terms as:

$$
\begin{aligned}
P^{-1}_{1,x}(\tilde{z}) &= \Re\left(\frac{4\tilde{z}}{1+\tilde{z}^4}\right) \equiv \Re(a) \\
P^{-1}_{1,y}(\tilde{z}) &= \Re\left(\frac{4i\tilde{z}}{1+\tilde{z}^4}\right) \equiv -\Im(a) \\
P^{-1}_{2,x}(\tilde{z}) &= \Im\left(\frac{4\tilde{z}}{1+\tilde{z}^4}\right) \equiv \Im(a) \\
P^{-1}_{2,y}(\tilde{z}) &= \Im\left(\frac{4i\tilde{z}}{1+\tilde{z}^4}\right) \equiv \Re(a) \\
P^{-1}_{1,x,x}(\tilde{z}) &= \Re\left(\frac{4(1-3\tilde{z}^4)}{(1+\tilde{z}^4)^2}\right) \equiv \Re(b) \\
P^{-1}_{1,x,y}(\tilde{z}) &= \Re\left(\frac{4i(1-3\tilde{z}^4)}{(1+\tilde{z}^4)^2}\right) \equiv -\Im(b) \\
P^{-1}_{2,x,x}(\tilde{z}) &= \Im\left(\frac{4(1-3\tilde{z}^4)}{(1+\tilde{z}^4)^2}\right) \equiv \Im(b) \\
P^{-1}_{2,x,y}(\tilde{z}) &= \Im\left(\frac{4i(1-3\tilde{z}^4)}{(1+\tilde{z}^4)^2}\right) \equiv \Re(b)
\end{aligned}
$$

Therefore the Hessians are

$$
HP_1^{-1} = \begin{pmatrix} \Re(b) & -\Im(b) \\ -\Im(b) & -\Re(b) \end{pmatrix}, \quad HP_2^{-1} = \begin{pmatrix} \Im(b) & \Re(b) \\ \Re(b) & -\Im(b) \end{pmatrix},
$$

Calculating further:

$$
\begin{aligned}
\tilde{z}_\alpha^T HP_2^{-1}\tilde{z}_\alpha &= \Re(b)(2\tilde{z}^1_\alpha\tilde{z}^2_\alpha) + \Im(b)((\tilde{z}^1_\alpha)^2 - (\tilde{z}^2_\alpha)^2) \\
\tilde{z}_\alpha^T HP_1^{-1}\tilde{z}_\alpha &= \Re(b)((\tilde{z}^1_\alpha)^2 - (\tilde{z}^2_\alpha)^2) - \Im(b)(2\tilde{z}^1_\alpha\tilde{z}^2_\alpha)
\end{aligned}
$$

$$
\begin{aligned}
X_1 &= \Re(a)\Re(b)(2(\tilde{z}^1_\alpha)^2\tilde{z}^2_\alpha) + \Re(a)\Im(b)((\tilde{z}^1_\alpha)^2\tilde{z}^1_\alpha - (\tilde{z}^2_\alpha)^2\tilde{z}^1_\alpha) \\
&\quad + \Im(a)\Re(b)(-2\tilde{z}^1_\alpha(\tilde{z}^2_\alpha)^2) + \Im(b)\Im(b)((\tilde{z}^1_\alpha)^2\tilde{z}^2_\alpha - (\tilde{z}^2_\alpha)^2\tilde{z}^2_\alpha) \\
X_2 &= \Re(b)\Re(b)((\tilde{z}^1_\alpha)^2\tilde{z}^2_\alpha - (\tilde{z}^2_\alpha)^2\tilde{z}^2_\alpha) + \Re(a)\Im(b)(-2\tilde{z}^1_\alpha(\tilde{z}^2_\alpha)^2) \\
&\quad + \Im(a)\Im(b)(2(\tilde{z}^1_\alpha)^2\tilde{z}^2_\alpha) + \Im(a)\Re(b)((\tilde{z}^1_\alpha)^2\tilde{z}^1_\alpha - (\tilde{z}^2_\alpha)^2\tilde{z}^1_\alpha)
\end{aligned}
$$

This means

$$
\begin{aligned}
X = X_1 - X_2 &= ((\tilde{z}^1_\alpha)^2 + (\tilde{z}^2_\alpha)^2)(\tilde{z}^2_\alpha(\Re(a)\Re(b) + \Im(a)\Im(b)) + \tilde{z}^1_\alpha(\Re(a)\Im(b) - \Im(a)\Re(b))) \\
&\equiv ((\tilde{z}^1_\alpha)^2 + (\tilde{z}^2_\alpha)^2)\langle G(z), \tilde{z}_\alpha\rangle.
\end{aligned}
$$

We can see that

$$
\begin{aligned}
-\frac{Q_\alpha}{Q^3} &= \frac{1}{2}\partial_\alpha\left(\frac{1}{Q^2}\right) = \partial_\alpha(\Re(a)^2 + \Im(a)^2) \\
&= \Re(a)\Re(b)\tilde{z}^1_\alpha - \Re(a)\Im(b)\tilde{z}^2_\alpha + \Im(a)\Im(b)\tilde{z}^1_\alpha + \Im(a)\Re(b)\tilde{z}^2_\alpha \\
&= \langle G(z), \tilde{z}^\perp_\alpha\rangle
\end{aligned}
$$

by the Cauchy-Riemann equations.
If we take one derivative in space of X, we obtain

$$
\begin{aligned}
\partial_\alpha X &= ((\tilde{z}^1_\alpha)^2 + (\tilde{z}^2_\alpha)^2)\langle \nabla G(\tilde{z})\cdot\tilde{z}_\alpha, \tilde{z}_\alpha\rangle + ((\tilde{z}^1_\alpha)^2 + (\tilde{z}^2_\alpha)^2)\langle G(\tilde{z}), \tilde{z}_{\alpha\alpha}\rangle \\
&= ((\tilde{z}^1_\alpha)^2 + (\tilde{z}^2_\alpha)^2)\langle \nabla G(\tilde{z})\cdot\tilde{z}_\alpha, \tilde{z}_\alpha\rangle + |\tilde{z}_\alpha|^3\tilde{K}\langle G(\tilde{z}), \tilde{z}^\perp_\alpha\rangle \\
&= ((\tilde{z}^1_\alpha)^2 + (\tilde{z}^2_\alpha)^2)\langle \nabla G(\tilde{z})\cdot\tilde{z}_\alpha, \tilde{z}_\alpha\rangle - |\tilde{z}_\alpha|^3\tilde{K}\frac{Q_\alpha}{Q^3},
\end{aligned}
$$

This implies

$$
K = Q\tilde{K} - Q^3\frac{X}{|\tilde{z}|^3} \Rightarrow K_\alpha = (Q\tilde{K})_\alpha + \frac{Q^3}{|\tilde{z}_\alpha|}\langle\nabla G(\tilde{z})\cdot\tilde{z}_\alpha, \tilde{z}_\alpha\rangle - \tilde{K}Q_\alpha = (Q\tilde{K})_\alpha + M_1 + M_2
$$

Later, we will see that the M_1 is a low order term and can be absorbed by the energy.

3.3 Initial data

For initial data we are interested in considering a self-intersecting curve in one point. More precisely, we will use as initial data *splash curves* which are defined this way:

Definition 3.3.1 *We say that $z(\alpha) = (z_1(\alpha), z_2(\alpha))$ is a* splash curve *if*

1. *$z_1(\alpha) - \alpha, z_2(\alpha)$ are smooth functions and 2π-periodic.*

2. *$z(\alpha)$ satisfies the arc-chord condition at every point except at α_1 and α_2, with $\alpha_1 < \alpha_2$ where $z(\alpha_1) = z(\alpha_2)$ and $|z_\alpha(\alpha_1)|, |z_\alpha(\alpha_2)| > 0$. This means $z(\alpha_1) = z(\alpha_2)$, but if we remove either a neighborhood of α_1 or a neighborhood of α_2 in parameter space, then the arc-chord condition holds.*

3. *The curve $z(\alpha)$ separates the complex plane into two regions; a connected water region and a vacuum region (not necessarily connected). The water region contains each point $x+iy$ for which y is large negative. We choose the parametrization such that the normal vector $n = \frac{(-\partial_\alpha z_2(\alpha), \partial_\alpha z_1(\alpha))}{|\partial_\alpha z(\alpha)|}$ points to the vacuum region. We regard the interface to be part of the water region.*

4. *We can choose a branch of the function P on the water region such that the curve $\tilde{z}(\alpha) = (\tilde{z}_1(\alpha), \tilde{z}_2(\alpha)) = P(z(\alpha))$ satisfies:*

(a) $\tilde{z}_1(\alpha)$ *and* $\tilde{z}_2(\alpha)$ *are smooth and* 2π*-periodic.*

(b) $\tilde{z}$ *is a closed contour.*

(c) $\tilde{z}$ *satisfies the arc-chord condition.*

We will choose the branch of the root that produces that

$$\lim_{y\to-\infty} P(x+iy) = -e^{-i\pi/4}$$

independently of x.

5. $P(w)$ *is analytic at* w *and* $\frac{dP}{dw}(w) \neq 0$ *if* w *belongs to the interior of the water region. Furthermore,* $(\pm\pi, 0)$ *and* $(0,0)$ *belong to the vacuum region.*

6. $\tilde{z}(\alpha) \neq q^l$ *for* $l = 0, ..., 4$, *where*

$$q^0 = (0,0)\,, \quad q^1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad q^2 = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad q^3 = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right), \quad q^4 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right). \tag{3.8}$$

Moreover, we will define a *splat curve* as a splash curve but replacing condition (2) by the fact that the curve touches itself along an arc, instead of a point.

Let us note that in order to measure when the transformation P is regular, we need to control the distance to the points q^l. In order to do so, we introduce the function

$$m(q^l)(\alpha, t) \equiv |\tilde{z}(\alpha, t) - q^l|$$

for $l = 0, \ldots, 4$.

We have performed numerical simulations, as explained in [19] with the following initial data on the non-tilde domain:

$$z_1^0(\alpha) = \alpha + \frac{1}{4}\left(-\frac{3\pi}{2} - 1.9\right)\sin(\alpha) + \frac{1}{2}\sin(2\alpha) + \frac{1}{4}\left(\frac{\pi}{2} - 1.9\right)\sin(3\alpha)$$

$$z_2^0(\alpha) = \frac{1}{10}\cos(\alpha) - \frac{3}{10}\cos(2\alpha) + \frac{1}{10}\cos(3\alpha)$$

Note that $z\left(\frac{\pi}{2}\right) = z\left(-\frac{\pi}{2}\right)$ (splash). Instead of prescribing an initial condition for $\tilde{\omega}$, we prescribed the normal component of the velocity to ensure a more controlled direction of the fluid. From that we got the initial $\tilde{\omega}(\alpha, 0)$ using the following relations. Let ψ be such that $\nabla^\perp\psi = v$ and $\Psi(\alpha)$ its restriction to the interface. The initial normal velocity is then prescribed by setting

$$u_n^0(\alpha)|z_\alpha(\alpha)| = \Psi_\alpha(\alpha) = 3\cdot\cos(\alpha) - 3.4\cdot\cos(2\alpha) + \cos(3\alpha) + 0.2\cos(4\alpha).$$

The reader may easily check that the above z_1^0 and z_2^0 yield a splash curve, i.e. the conditions in Definition 3.3.1 are satisfied. See Figure 3.1.

In order to get an initial data for the splat singularity, one only needs to perturb the splash curve so that it $z_0^1(\alpha) = 0$ on a neighbourhood of both $\alpha = \pm\frac{\pi}{2}$. The normal velocity

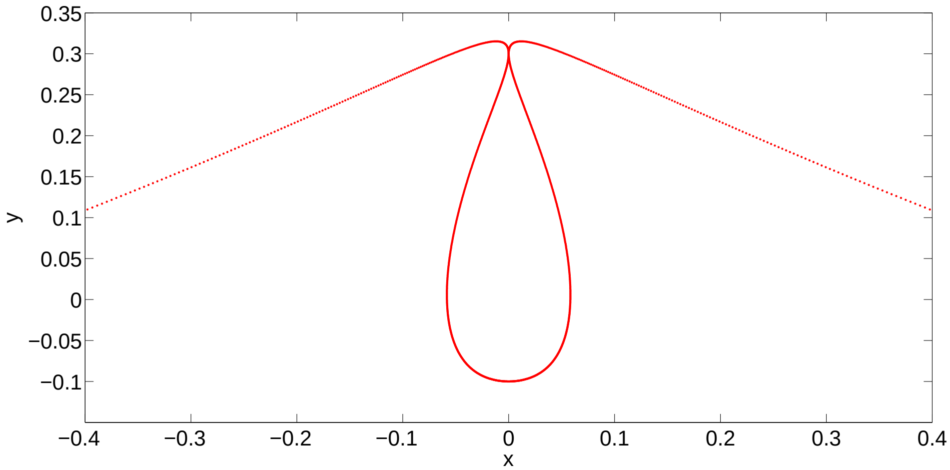


Figure 3.1: Splash singularity. The interface self intersects in a point.

can be the same since it has the right sign (the one that separates the curve). By continuity, the Rayleigh-Taylor function should remain positive.

For the case where the energy is independent on the surface tension coefficient (see Section 3.5), we need the curve to satisfy the Rayleigh-Taylor condition initially. This is always the case when the surface tension coefficient is small enough. To illustrate this phenomenon, we have plotted in the next figure the Rayleigh-Taylor condition for different values of the surface tension coefficient and the initial condition described above. We can see that for small enough values of τ (0 and 0.1): the Rayleigh-Taylor condition σ is strictly positive. For bigger values of τ, the Rayleigh-Taylor condition σ has distinct sign.

3.4 Energy without the Rayleigh-Taylor condition

In this section, we prove local existence in the tilde domain, where the time of existence depends on the surface tension coefficient. This theorem has the advantage that the initial data does not need to satisfy the Rayleigh-Taylor condition and it works for every $\tau > 0$.

Theorem 3.4.1 *Let $k \geq 3$. Let $\tilde{z}^0(\alpha)$ be the image of a splash curve by the map P parametrized in such a way that $|\partial_\alpha \tilde{z}^0(\alpha)| = \frac{L}{2\pi}$, where L is the length of the curve in a fundamental period, and such that $\tilde{z}_1^0(\alpha), \tilde{z}_2^0(\alpha) \in H^{k+2}(\mathbb{T})$. Let $\tilde{\omega}(\alpha, 0) \in H^{k+\frac{1}{2}}(\mathbb{T})$. Then there exist a finite time $T > 0$, a time-varying curve $\tilde{z}(\alpha, t) \in C([0, T]; H^{k+2})$, and a function $\tilde{\omega}(\alpha, t) \in C([0, T]; H^{k+\frac{1}{2}})$ providing a solution of the water wave equations (3.1 - 3.2).*

The proof below is based in the following energy estimates:

3.4.1 The energy

We will define the energy for $k \geq 3$ as

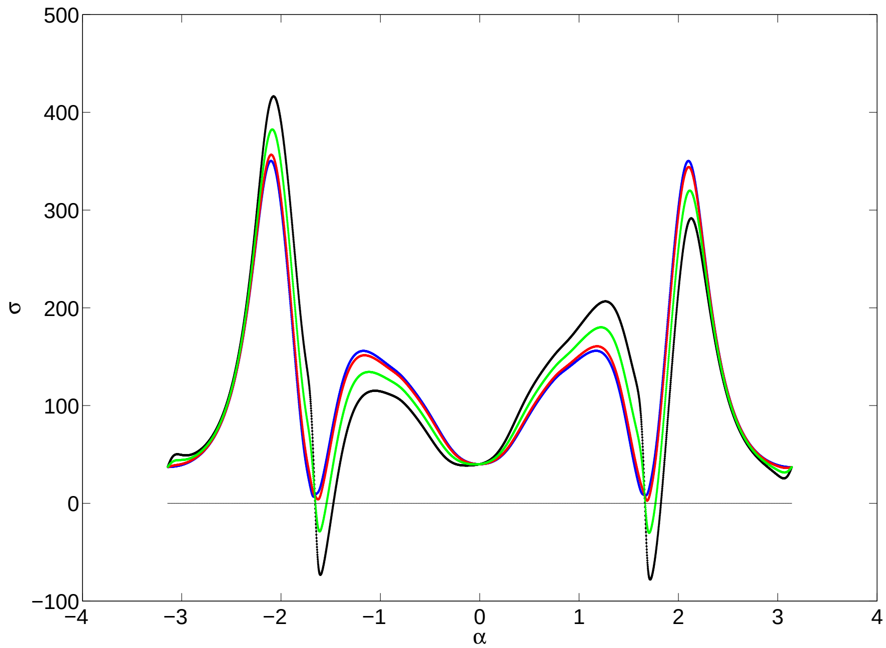


Figure 3.2: Rayleigh-Taylor function for different values of τ: $\tau = 0$ (blue), $\tau = 0.1$ (red), $\tau = 0.5$ (green), $\tau = 1$ (black)

$$E_k^2(t) = \mathcal{E}\mathcal{E}^2(t) + \underbrace{2|\tilde{z}_\alpha|^3 \int Q^{2k+1} \left(\partial_\alpha^k(\tilde{K})\right)^2}_{A} + \underbrace{\frac{1}{\tau}\int Q^{2k+2}\partial_\alpha^k(\tilde{\omega})\Lambda(\partial_\alpha^k(\tilde{\omega}))}_{B} + \underbrace{\frac{1}{2|\tilde{z}_\alpha|\tau^2}\int Q^{2k+3}(\partial_\alpha^k(\tilde{\omega}))^2\tilde{\omega}^2}_{C},$$

$$\mathcal{E}\mathcal{E}^2(t) = \|\tilde{z}\|_{L^2}^2 + \|\tilde{\omega}\|_{L^2}^2 + \|\mathcal{F}(z)\|_{L^\infty}^2(t) + \sum_{l=0}^{4} \frac{1}{m(q^l)(t)},$$

where $m(q^l)(t) = \min_{\alpha\in\mathbb{T}} q^l(\alpha, t)$ for $l = 0, \ldots, 4$ and $\Lambda = (-\Delta)^{1/2}$. From now on, we will denote the Hilbert transform of a function f by $H(f)$, where

$$H(f)(\alpha) = \frac{PV}{\pi}\int_{-\pi}^{\pi} \frac{f(\alpha-\beta)}{2\tan\left(\frac{\beta}{2}\right)} d\beta.$$

Recall that the operator Λ can also be written as $\Lambda(f) = \partial_\alpha H(f)$.

3.4.2 The energy estimates

The energy estimates for $\mathcal{E}\mathcal{E}$ were proved in [28] and in [20]. In this section we will focus on the new terms (A, B and C).

3.4.2.1 $\tilde{K}$

Proposition 3.4.2

$$\tilde{K}_t = NICE3\ + \frac{Q^2}{2|\tilde{z}_\alpha|^3}H(\tilde{\omega}_{\alpha\alpha}) + \frac{1}{|\tilde{z}_\alpha|^3}(Q^2)_\alpha H(\tilde{\omega}_\alpha),$$

where NICE3 means

$$\int Q^j \partial_\alpha^k(\tilde{K})\partial_\alpha^k(NICE3) \leq CE_k^p(t)$$

for some positive constants C, p and any j.

Proof: We start writing $\tilde{K}_t$

$$\tilde{K}_t = \frac{-3}{|\tilde{z}_\alpha|^5}\tilde{z}_{\alpha t}\cdot\tilde{z}_\alpha\tilde{z}_{\alpha\alpha}\cdot\tilde{z}_\alpha^\perp + \frac{1}{|\tilde{z}_\alpha|^3}\left(\tilde{z}_{\alpha\alpha t}\cdot\tilde{z}_\alpha^\perp + \tilde{z}_{\alpha\alpha}\cdot\tilde{z}_{\alpha t}^\perp\right) = P_0 + P_1 + P_2$$

Calculating further P_0 we get that

$$P_0 = \frac{-3}{|\tilde{z}_\alpha|^5}(Q^2BR + c\tilde{z}_\alpha)_\alpha\cdot\tilde{z}_\alpha\tilde{z}_{\alpha\alpha}\cdot\tilde{z}_\alpha^\perp = \text{NICE3},$$

by the estimates proved in the Appendix.

On the one hand, developing P_2, we obtain

$$\begin{aligned} P_2 &= \frac{1}{|\tilde{z}_\alpha|^3}\left(\tilde{z}_{\alpha\alpha}\cdot\tilde{z}_{\alpha t}^\perp\right) = -\frac{1}{|\tilde{z}_\alpha|^3}\left(\tilde{z}_{\alpha\alpha}^\perp\cdot\tilde{z}_{\alpha t}\right) = \\ &= -\frac{1}{|\tilde{z}_\alpha|^3}\left((Q^2BR)_\alpha + (\tilde{c}\tilde{z}_\alpha)_\alpha\right)\cdot\tilde{z}_{\alpha\alpha}^\perp \\ &= \text{NICE3}\ - \frac{1}{|\tilde{z}_\alpha|^3}\left(\frac{Q^2}{2}\frac{\tilde{z}_\alpha^\perp}{|\tilde{z}_\alpha|^2}H(\tilde{\omega}_\alpha) + \tilde{c}_\alpha\tilde{z}_\alpha\right)\cdot\tilde{z}_{\alpha\alpha}^\perp = \text{NICE3}, \end{aligned}$$

since $\tilde{c}_\alpha$ is as regular as $\tilde{\omega}, \tilde{z}_{\alpha\alpha}$ and therefore bounded in H^k. On the other, P_1 gives rise to

$$P_1 = \frac{1}{|\tilde{z}_\alpha|^3}\left(\tilde{z}_{\alpha\alpha t}\cdot\tilde{z}_\alpha^\perp\right) = \frac{1}{|\tilde{z}_\alpha|^3}\left((Q^2BR)_{\alpha\alpha} + (\tilde{c}\tilde{z}_\alpha)_{\alpha\alpha}\right)\cdot\tilde{z}_\alpha^\perp = P_{1,1} + P_{1,2}$$

We can further develop $P_{1,2}$ to obtain

$$P_{1,2} = \text{NICE3},$$

since the terms vanish either by integrating by parts, by being a dot product between two orthogonal vectors or because $\tilde{c}_\alpha = \text{NICE3}$. We also have that

$$P_{1,1} = \text{NICE3}\ + \frac{1}{|\tilde{z}_\alpha|^3}\left(2(Q^2)_\alpha BR_\alpha + Q^2BR_{\alpha\alpha}\right)\cdot\tilde{z}_\alpha^\perp = \text{NICE3}\ + P_{1,1,1} + P_{1,1,2}$$

The only term in BR_α which is not NICE3 is when we hit with the derivative in $\tilde{\omega}$. Therefore

$$P_{1,1,1} = \text{NICE3 } + \frac{1}{|\tilde{z}_\alpha|^3} 2(Q^2)_\alpha \frac{1}{2} H(\tilde{\omega}_\alpha)$$

Finally, regarding $P_{1,1,2}$ and keeping in mind that hitting with all the derivatives in z leads us to a term which has the factor $\tilde{z}_{\alpha\alpha\alpha} \cdot \tilde{z}_\alpha = -|\tilde{z}_{\alpha\alpha}|^2$, giving us the extra regularity we needed to integrate the term.

$$\begin{aligned} P_{1,1,2} &= \text{NICE3 } + \frac{Q^2}{|\tilde{z}_\alpha|^3} \tilde{z}_\alpha^\perp \cdot \left(\frac{1}{2\pi} \int \frac{(\tilde{z}(\alpha) - \tilde{z}(\beta))^\perp}{|\tilde{z}(\alpha) - \tilde{z}(\beta)|^2} \tilde{\omega}_{\alpha\alpha}(\beta) d\beta \right) \\ &= \text{NICE3 } + \frac{Q^2}{2|\tilde{z}_\alpha|^3} H(\tilde{\omega}_{\alpha\alpha}). \end{aligned}$$

We should notice that there doesn't appear a term proportional to $H(\tilde{\omega}_\alpha)$ since the kernel that results from subtracting the Hilbert transform has room for two derivatives instead of one.

Adding all the previous estimates together we get the desired result. □

3.4.2.2 $\tilde{\omega}$

We first notice that M_1 (one of the terms in the curvature) is of the order of z_α and therefore it can be absorbed by the energy. Hence

$$\partial_\alpha K = (\tilde{K} Q)_\alpha - \tilde{K} Q_\alpha + \text{low order terms}$$

We will follow the proof done by Ambrose in [6]. Taking into account the estimates for the implicit operator done in [28], we are left to see the impact of the Q factor in the singular term $(\tilde{c}\tilde{\omega})_\alpha$, since the impact into the others is either trivial (the ones that come from the factor proportional to the curvature) or is zero (the rest of the terms).

Lemma 3.4.3

$$\partial_\alpha^k (\tilde{c}_\alpha \tilde{\omega}) = \mathit{NICE35} \ + \frac{Q^2 \tilde{\omega}^2}{2|\tilde{z}_\alpha|} H(\partial_\alpha^k(\tilde{K})),$$

where NICE35 means

$$\int Q^j \Lambda(\partial_\alpha^k(\tilde{\omega})) \mathit{NICE35} \le C E_k^p(t)$$

for some positive constants C, p and any j.

Proof: The most singular term is when we hit all the derivatives in $\tilde{c}_\alpha$, since if we hit all of them in $\tilde{\omega}$, that term would belong to NICE35. Developing the new terms

$$\begin{aligned}
\partial_\alpha^k(\tilde{c}_\alpha\tilde{\omega}) &= \text{NICE35} \ -\tilde{\omega}\partial_\alpha^k\left((Q^2BR)_\alpha\cdot\frac{\tilde{z}_\alpha}{|\tilde{z}_\alpha|^2}\right)\\
&= \text{NICE35} \ -\frac{Q^2\tilde{\omega}}{|\tilde{z}_\alpha|^2}\cdot\partial_\alpha^k\left(\tilde{z}_\alpha\cdot\int\frac{(\tilde{z}_\alpha(\alpha)-\tilde{z}_\alpha(\beta))^\perp}{|\tilde{z}(\alpha)-\tilde{z}(\beta)|^2}\tilde{\omega}(\beta)d\beta\right)\\
&= \text{NICE35} \ -\frac{Q^2\tilde{\omega}^2}{2|\tilde{z}_\alpha|^4}\partial_\alpha^k\left(\tilde{z}_\alpha\cdot H(\tilde{z}_{\alpha\alpha}^\perp)\right)\\
&= \text{NICE35} \ +\frac{Q^2\tilde{\omega}^2}{2|\tilde{z}_\alpha|}H\left(\partial_\alpha^k(\tilde{K})\right).
\end{aligned}$$

□

Lemma 3.4.4

$$\partial_\alpha^k(\tilde{c}\tilde{\omega}_\alpha) = \mathit{NICE35}\ ,$$

where NICE35 means

$$\int Q^j\Lambda(\partial_\alpha^k(\tilde{\omega}))\mathit{NICE35}\le CE_k^p(t)$$

for some positive constants C, p *and any* j.

Proof: The most singular term is when we hit all the derivatives in $\tilde{\omega}_\alpha$, since if we hit all of them in $\tilde{c}$, that term would belong to NICE35. Thus, we have to estimate

$$\begin{aligned}
\int Q^jH\partial_\alpha^{k+1}(\tilde{\omega})\partial_\alpha^{k+1}(\tilde{\omega})\tilde{c} &= -\int\partial_\alpha^{k+1}(\tilde{\omega})H(\partial_\alpha^{k+1}(\tilde{\omega})Q^j\tilde{c})\\
&= \frac{1}{2}\int\partial_\alpha^{k+1}(\tilde{\omega})\left[H(\partial_\alpha^{k+1}(\tilde{\omega}))\tilde{c}Q^j - H(\partial_\alpha^{k+1}(\tilde{\omega})\tilde{c}Q^j)\right]\le CE_k^p(t),
\end{aligned}$$

and therefore it is NICE35. □

3.4.3 Calculations of the time derivative of the energy

Using the previous lemmas and propositions, we can get the following estimates for the derivative of the energy:

$$\begin{aligned}
\frac{dA}{dt} &= \text{OK} \ +2\int Q^{2k+1}\left(\partial_\alpha^k(\tilde{K})\right)\partial_\alpha^k\left(Q^2H(\tilde{\omega}_{\alpha\alpha})+4QQ_\alpha H(\tilde{\omega}_\alpha)\right)\\
&= \text{OK} \ +2\int 2kQ^{2k+2}Q_\alpha\left(\partial_\alpha^k(\tilde{K})\right)\partial_\alpha^{k-1}\left(H(\tilde{\omega}_{\alpha\alpha})\right)\\
&+2\int Q^{2k+3}\left(\partial_\alpha^k(\tilde{K})\right)\partial_\alpha^k\left(H(\tilde{\omega}_{\alpha\alpha})\right)\\
&+2\int 4Q^{2k+2}Q_\alpha\left(\partial_\alpha^k(\tilde{K})\right)\partial_\alpha^k\left(H(\tilde{\omega}_\alpha)\right)\\
&= A^1+A^2+A^3,
\end{aligned}$$

where we will say that a term is OK if it is controlled by the energy.
We should be careful while estimating B_t because

$$\begin{aligned}\frac{dB}{dt} &= \text{OK } + \frac{1}{\tau}\int Q^{2k+2}\partial_\alpha^k(\tilde{\omega}_t)\Lambda(\partial_\alpha^k(\tilde{\omega})) + \frac{1}{\tau}\int Q^{2k+2}\partial_\alpha^k(\tilde{\omega})\Lambda(\partial_\alpha^k(\tilde{\omega}_t))\\ &= \text{OK } + \frac{1}{\tau}\int Q^{2k+2}\partial_\alpha^k(\tilde{\omega}_t)\Lambda(\partial_\alpha^k(\tilde{\omega})) + \frac{1}{\tau}\int \Lambda(Q^{2k+2}\partial_\alpha^k(\tilde{\omega}))\partial_\alpha^k(\tilde{\omega}_t)\\ &= \text{OK } + \frac{2}{\tau}\int Q^{2k+2}\partial_\alpha^k(\tilde{\omega}_t)\Lambda(\partial_\alpha^k(\tilde{\omega})) + \frac{1}{\tau}\int (Q^{2k+2})_\alpha H(\partial_\alpha^k(\tilde{\omega}))\partial_\alpha^k(\tilde{\omega}_t)\end{aligned}$$

Hence

$$\begin{aligned}\frac{dB}{dt} &= \text{OK } + \frac{2}{\tau}\int Q^{2k+2}\Lambda(\partial_\alpha^k(\tilde{\omega}))\frac{Q^2\tilde{\omega}^2}{2|\tilde{z}_\alpha|}H(\partial_\alpha^k(\tilde{K}))\\ &+ 2\int Q^{2k+2}\Lambda(\partial_\alpha^k(\tilde{\omega}))\partial_\alpha^k((Q\tilde{K} + \frac{Q^3}{|\tilde{z}_\alpha|^3}X)_\alpha)\\ &+ \int (2k+2)Q^{2k+1}Q_\alpha H(\partial_\alpha^k(\tilde{\omega}))\partial_\alpha^{k+1}((\tilde{K}Q))\\ &= \text{OK } + \frac{2}{\tau}\int Q^{2k+2}\Lambda(\partial_\alpha^k(\tilde{\omega}))\frac{Q^2\tilde{\omega}^2}{2|\tilde{z}_\alpha|}H(\partial_\alpha^k(\tilde{K}))\\ &+ 2\int Q^{2k+2}\Lambda(\partial_\alpha^k(\tilde{\omega}))\partial_\alpha^k((Q\tilde{K})_\alpha) - 2\int Q^{2k+2}Q_\alpha\Lambda(\partial_\alpha^k(\tilde{\omega}))\partial_\alpha^k(\tilde{K})\\ &+ \int (2k+2)Q^{2k+2}Q_\alpha H(\partial_\alpha^k(\tilde{\omega}))\partial_\alpha^{k+1}(\tilde{K})\\ &= \text{OK } + B^1 + B^2 + B^3 + B^4\end{aligned}$$

$$\frac{dC}{dt} = \text{OK } + \frac{1}{|\tilde{z}_\alpha|\tau}\int Q^{2k+4}\tilde{\omega}^2\partial_\alpha^k(\tilde{\omega})\partial_\alpha^{k+1}(\tilde{K}) = \text{OK } + C^1$$

3.4.4 Development of the derivative in B

We start from the development of B^1, B^2, B^3 and B^4. We trivially have:

$$\begin{aligned}B^1 &= \frac{1}{\tau}\int Q^{2k+2}\Lambda(\partial_\alpha^k(\tilde{\omega}))\frac{Q^2\tilde{\omega}^2}{|\tilde{z}_\alpha|}H(\partial_\alpha^k(\tilde{K}))\\ B^3 &= -2\int Q^{2k+2}Q_\alpha\Lambda(\partial_\alpha^k(\tilde{\omega}))\partial_\alpha^k(\tilde{K})\\ B^4 &= \text{OK } - \int (2k+2)Q^{2k+2}Q_\alpha H(\partial_\alpha^{k+1}(\tilde{\omega}))\partial_\alpha^k(\tilde{K})\end{aligned}$$

We now look at B^2. We can decompose it in the following way

$$
\begin{aligned}
B^2 &= 2\int Q^{2k+2}\Lambda(\partial_\alpha^k(\tilde{\omega}))\partial_\alpha^k(Q_\alpha\tilde{K}+Q\tilde{K}_\alpha)\\
&= \mathrm{OK}\ +2\int Q^{2k+2}\Lambda(\partial_\alpha^k(\tilde{\omega}))(Q_\alpha\partial_\alpha^k(\tilde{K})+Q\partial_\alpha^{k+1}(\tilde{K})+kQ_\alpha\partial_\alpha^k(\tilde{K}))\\
&= \mathrm{OK}\ +B^{2,1}+B^{2,2}+B^{2,3}
\end{aligned}
$$

We can write down the terms $B^{2,1}$ and $B^{2,3}$ in the form

$$
\begin{aligned}
B^{2,1} &= 2\int Q^{2k+2}H(\partial_\alpha^{k+1}(\tilde{\omega}))Q_\alpha\partial_\alpha^k(\tilde{K})\\
B^{2,3} &= 2k\int Q^{2k+2}H(\partial_\alpha^{k+1}(\tilde{\omega}))Q_\alpha\partial_\alpha^k(\tilde{K})
\end{aligned}
$$

Integrating by parts in $B^{2,2}$ we establish

$$
\begin{aligned}
B^{2,2} &= -2\int Q^{2k+3}\Lambda(\partial_\alpha^{k+1}(\tilde{\omega}))\partial_\alpha^k(\tilde{K})\\
&\quad -2(2k+3)\int Q^{2k+2}Q_\alpha\Lambda(\partial_\alpha^k(\tilde{\omega}))\partial_\alpha^k(\tilde{K})\\
&= B^{2,2,1}+B^{2,2,2}
\end{aligned}
$$

Again, $B^{2,2,2}$ can easily be reduced to the canonical form

$$
B^{2,2,2} = -2(2k+3)\int Q^{2k+2}Q_\alpha H(\partial_\alpha^{k+1}(\tilde{\omega}))\partial_\alpha^k(\tilde{K})
$$

3.4.5 Collection of the terms

We will split all the uncontrolled terms into three categories: high order and low order types I and II and we will see that the sum of the terms in each category adds up to low enough order terms, denoted by OK.

3.4.5.1 High Order

From A:

$$
2\int Q^{2k+3}\left(\partial_\alpha^k(\tilde{K})\right)\partial_\alpha^k\left(H(\tilde{\omega}_{\alpha\alpha})\right)\quad (A^2)
$$

From B:

$$
-2\int Q^{2k+3}\Lambda(\partial_\alpha^{k+1}(\tilde{\omega}))\partial_\alpha^k(\tilde{K})\quad (B^{2,2,1})
$$

From C:

No terms from C.

3.4.5.2 Low Order Type I

From A:

$$2\int 2kQ^{2k+2}Q_\alpha\left(\partial_\alpha^k(\tilde{K})\right)\partial_\alpha^{k-1}\left(H(\tilde{\omega}_{\alpha\alpha})\right) \quad (A^1)$$

$$2\int 4Q^{2k+2}Q_\alpha\left(\partial_\alpha^k(\tilde{K})\right)\partial_\alpha^{k}\left(H(\tilde{\omega}_{\alpha})\right) \quad (A^3)$$

From B:

$$-2\int Q^{2k+2}Q_\alpha\Lambda(\partial_\alpha^k(\tilde{\omega}))\partial_\alpha^k(\tilde{K}) \quad (B^3)$$

$$-\int(2k+2)Q^{2k+2}Q_\alpha H(\partial_\alpha^{k+1}(\tilde{\omega}))\partial_\alpha^k(\tilde{K}) \quad (B^4)$$

$$2\int Q^{2k+2}H(\partial_\alpha^{k+1}(\tilde{\omega}))Q_\alpha\partial_\alpha^k(\tilde{K})) \quad (B^{2,1})$$

$$2k\int Q^{2k+2}H(\partial_\alpha^{k+1}(\tilde{\omega}))Q_\alpha\partial_\alpha^k(\tilde{K}) \quad (B^{2,3})$$

$$-2(2k+3)\int Q^{2k+2}Q_\alpha H(\partial_\alpha^{k+1}(\tilde{\omega}))\partial_\alpha^k(\tilde{K}) \quad (B^{2,2,2})$$

From C:

No terms from C.

3.4.5.3 Low Order Type II

From A:

No terms from A.

From B:

$$\frac{1}{\tau}\int Q^{2k+2}\Lambda(\partial_\alpha^k(\tilde{\omega}))\frac{Q^2\tilde{\omega}^2}{|\tilde{z}_\alpha|}H(\partial_\alpha^k(\tilde{K})) \quad (B^1)$$

From C:

$$\frac{1}{|\tilde{z}_\alpha|\tau}\int Q^{2k+4}\tilde{\omega}^2\partial_\alpha^k(\tilde{\omega})\partial_\alpha^{k+1}(\tilde{K}) \quad (C^1)$$

3.4.6 Regularized system

Now, let $\tilde{z}^{\varepsilon,\delta,\mu}(\alpha,t)$ be a solution of the following system (compare with (3.1 - 3.2)):

$$\tilde{z}_t^{\varepsilon,\delta,\mu}(\alpha,t)=\phi_\delta*\phi_\delta*\left(Q^2(\tilde{z}^{\varepsilon,\delta,\mu})BR(\tilde{z}^{\varepsilon,\delta,\mu},\tilde{\omega}^{\varepsilon,\delta,\mu})\right)(\alpha,t)+\phi_\mu*\left(\tilde{c}^{\varepsilon,\delta,\mu}\left(\phi_\mu*\partial_\alpha\tilde{z}^{\varepsilon,\delta,\mu}\right)\right)(\alpha,t), \tag{3.9}$$

$$
\begin{aligned}
\tilde{\omega}_t^{\varepsilon,\delta,\mu} = &\phi_\delta * \phi_\delta * \Big(-2BR_t(\tilde{z}^{\varepsilon,\delta,\mu},\tilde{\omega}^{\varepsilon,\delta,\mu})\cdot\tilde{z}_\alpha^{\varepsilon,\delta,\mu} - |BR(\tilde{z}^{\varepsilon,\delta,\mu},\tilde{\omega}^{\varepsilon,\delta,\mu})|^2(Q^2(\tilde{z}^{\varepsilon\delta,\mu}))_\alpha \\
&- \Big(\frac{Q^2(\omega^{\varepsilon,\tilde{\delta},\mu})^2}{4|\tilde{z}_\alpha^{\varepsilon,\delta,\mu}|^2}\Big)_\alpha + 2\tilde{c}^{\varepsilon,\delta,\mu}BR_\alpha(\tilde{z}^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu})\cdot\tilde{z}_\alpha^{\varepsilon,\delta,\mu} + \Big(\overline{c}^{\varepsilon,\delta,\mu}\tilde{\omega}^{\varepsilon,\delta,\mu}\Big)_\alpha - 2\left(P_2^{-1}(\tilde{z}^{\varepsilon,\delta,\mu}(\alpha,t))\right)_\alpha \\
&+\tau\left(\frac{Q^3(\tilde{z}^{\varepsilon,\delta,\mu})}{|\tilde{z}_\alpha^{\varepsilon,\delta,\mu}(\alpha,t)|^3}(\tilde{z}_\alpha^{\varepsilon,\delta,\mu})^THP_2^{-1}\tilde{z}_\alpha^{\varepsilon,\delta,\mu}\nabla P_1^{-1}\cdot\tilde{z}_\alpha^{\varepsilon,\delta,\mu} - (\tilde{z}_\alpha^{\varepsilon,\delta,\mu})^THP_1^{-1}\tilde{z}_\alpha^{\varepsilon,\delta,\mu}\nabla P_2^{-1}\cdot\tilde{z}_\alpha^{\varepsilon,\delta,\mu})\right)_\alpha \\
&+\tau\left(Q\frac{\tilde{z}_{\alpha\alpha}^{\varepsilon,\delta,\mu}\cdot(\tilde{z}_\alpha^{\varepsilon,\delta,\mu})^\perp}{|\tilde{z}_\alpha^{\varepsilon,\delta,\mu}|^3}\right)_\alpha\Bigg) - \varepsilon\phi_\mu*\phi_\mu*\left(\Lambda(\tilde{\omega}^{\varepsilon,\delta,\mu})\frac{1}{Q^{2k+3}}\right)
\end{aligned} \tag{3.10}
$$

$\tilde{z}^{\varepsilon,\delta,\mu}(\alpha,0)=\tilde{z}_0(\alpha)$ and $\tilde{\omega}^{\varepsilon,\delta,\mu}(\alpha,0)=\tilde{\omega}_0(\alpha)$ for $\varepsilon>0$, $\delta>0,\mu>0$. The functions ϕ_δ and ϕ_μ are even mollifiers,

$$
\begin{aligned}
\tilde{c}^{\varepsilon,\delta,\mu}(\alpha) =&\frac{\alpha+\pi}{2\pi}\int_{-\pi}^{\pi}\frac{\partial_\beta\tilde{z}^{\varepsilon,\delta,\mu}(\beta))}{|\partial_\beta\tilde{z}^{\varepsilon,\delta,\mu}(\beta)|^2}\cdot\phi_\delta*\phi_\delta*(\partial_\beta(Q^2(\tilde{z}^{\varepsilon,\delta,\mu})(\beta)BR(\tilde{z}^{\varepsilon,\delta,\mu},\tilde{\omega}^{\varepsilon,\delta,\mu}))(\beta))d\beta \\
&-\int_{-\pi}^{\alpha}\frac{\partial_\beta\tilde{z}^{\varepsilon,\delta,\mu}(\beta)}{|\partial_\beta\tilde{z}^{\varepsilon,\delta,\mu}(\beta)|^2}\cdot\phi_\delta*\phi_\delta*(\partial_\beta(Q^2(\tilde{z}^{\varepsilon,\delta,\mu})(\beta)BR(\tilde{z}^{\varepsilon,\delta,\mu},\tilde{\omega}^{\varepsilon,\delta,\mu}))(\beta))d\beta,
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{c}^{\varepsilon,\delta,\mu}(\alpha) =&\frac{\alpha+\pi}{2\pi}\int_{-\pi}^{\pi}\frac{\partial_\beta\tilde{z}^{\varepsilon,\delta,\mu}(\beta))}{|\partial_\beta\tilde{z}^{\varepsilon,\delta,\mu}(\beta)|^2}\cdot(\partial_\beta(Q^2(\tilde{z}^{\varepsilon,\delta,\mu})(\beta)BR(\tilde{z}^{\varepsilon,\delta,\mu},\tilde{\omega}^{\varepsilon,\delta,\mu}))(\beta))d\beta \\
&-\int_{-\pi}^{\alpha}\frac{\partial_\beta\tilde{z}^{\varepsilon,\delta,\mu}(\beta)}{|\partial_\beta\tilde{z}^{\varepsilon,\delta,\mu}(\beta)|^2}\cdot(\partial_\beta(Q^2(\tilde{z}^{\varepsilon,\delta,\mu})(\beta)BR(\tilde{z}^{\varepsilon,\delta,\mu},\tilde{\omega}^{\varepsilon,\delta,\mu}))(\beta))d\beta,
\end{aligned}
$$

The RHS of the evolution equations for $\tilde{z}^{\varepsilon,\delta,\mu}$ and $\tilde{\omega}^{\varepsilon,\delta,\mu}$ are Lipschitz in the spaces $H^{k+2}(\mathbb{T})$ and $H^{k+\frac{1}{2}}(\mathbb{T})$ since they are mollified. Therefore we can solve (3.9-3.10) for short time, thanks to Picard's theorem.

Now, we can perform energy estimates to get uniform bounds in μ (we just deal with a transport term and a dissipative) and we can let μ go to zero. The energy estimates that we can get are the following:

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\tilde{z}^{\varepsilon,\delta,\mu}\|_{H^5}^2+\|\mathcal{F}(\tilde{z}^{\varepsilon,\delta,\mu})\|_{L^\infty}^2+\|\tilde{\omega}^{\varepsilon,\delta,\mu}\|_{H^{3+\frac{1}{2}}}^2+\sum_{l=0}^{4}\frac{1}{m^{\varepsilon,\delta,\mu}(q^l)}\right)(t) \\
&\leq C(\delta)\left(\|\tilde{z}^{\varepsilon,\delta,\mu}\|_{H^5}^2+\|\mathcal{F}(\tilde{z}^{\varepsilon,\delta,\mu})\|_{L^\infty}^2+\|\tilde{\omega}^{\varepsilon,\delta,\mu}\|_{H^{3+\frac{1}{2}}}^2+\sum_{l=0}^{4}\frac{1}{m^{\varepsilon,\delta,\mu}(q^l)}\right)^j(t).
\end{aligned}
$$

We should note that for the new system without the ϕ_μ mollifier, the length of the tangent vector $|\partial_\alpha\tilde{z}^\delta|$ is now constant in space and depends only on time. Next we will perform energy estimates as in the previous case by using the curvature $\tilde{K}^\delta$ from the curve $\tilde{z}^\delta$.

Similarly, we get (let us omit the superscript δ,ε in $\tilde{z}^{\delta,\varepsilon}$ and $\tilde{\omega}^{\delta,\varepsilon}$)

- $$\tilde{K}_t = \text{NICE3 } + \frac{Q^2}{2|\tilde{z}_\alpha|^3}\phi_\delta * \phi_\delta * H(\tilde{\omega}_{\alpha\alpha}) + \frac{1}{|\tilde{z}_\alpha|^3}(Q^2)_\alpha \phi_\delta * \phi_\delta * H(\tilde{\omega}_\alpha),$$

- $$\partial_\alpha^k(\overline{c}_\alpha \tilde{\omega}) = \text{NICE35 } + \frac{Q^2\tilde{\omega}^2}{2|\tilde{z}_\alpha|}H(\partial_\alpha^k(\tilde{K})),$$

- $$\partial_\alpha^k(\tilde{c}\tilde{\omega}_\alpha) = \text{NICE35 },$$

and the following collection of terms:

3.4.6.1 High Order

From A:

$$2\int Q^{2k+3}\left(\partial_\alpha^k(\tilde{K})\right)\partial_\alpha^k \phi_\delta * \phi_\delta * (H(\tilde{\omega}_{\alpha\alpha})) \quad (A^2)$$

From B:

$$-2\int Q^{2k+3}\Lambda(\partial_\alpha^{k+1}(\tilde{\omega}))\phi_\delta * \phi_\delta * \partial_\alpha^k(\tilde{K}) \quad (B^{2,2,1})$$
$$-2\frac{\varepsilon}{\tau}\|\partial_\alpha^{k+1}\tilde{\omega}\|_{L^2}^2 \quad (D)$$

From C:

No terms from C.

3.4.6.2 Low Order Type I

From A:

$$2\int 2kQ^{2k+2}Q_\alpha\left(\partial_\alpha^k(\tilde{K})\right)\partial_\alpha^{k-1}\phi_\delta * \phi_\delta * (H(\tilde{\omega}_{\alpha\alpha})) \quad (A^1)$$
$$2\int 4Q^{2k+2}Q_\alpha\left(\partial_\alpha^k(\tilde{K})\right)\phi_\delta * \phi_\delta * \partial_\alpha^k(H(\tilde{\omega}_\alpha)) \quad (A^3)$$

From B:

$$-2\int Q^{2k+2}Q_\alpha\Lambda(\partial_\alpha^k(\tilde{\omega}))\phi_\delta * \phi_\delta * \partial_\alpha^k(\tilde{K}) \quad (B^3)$$
$$-\int(2k+2)Q^{2k+2}Q_\alpha H(\partial_\alpha^{k+1}(\tilde{\omega}))\phi_\delta * \phi_\delta * \partial_\alpha^k(\tilde{K}) \quad (B^4)$$
$$2\int Q^{2k+2}H(\partial_\alpha^{k+1}(\tilde{\omega}))Q_\alpha\phi_\delta * \phi_\delta * \partial_\alpha^k(\tilde{K})) \quad (B^{2,1})$$
$$2k\int Q^{2k+2}H(\partial_\alpha^{k+1}(\tilde{\omega}))Q_\alpha\phi_\delta * \phi_\delta * \partial_\alpha^k(\tilde{K}) \quad (B^{2,3})$$
$$-2(2k+3)\int Q^{2k+2}Q_\alpha H(\partial_\alpha^{k+1}(\tilde{\omega}))\phi_\delta * \phi_\delta * \partial_\alpha^k(\tilde{K}) \quad (B^{2,2,2})$$

From C:

No terms from C.

3.4.6.3 Low Order Type II

From A:
No terms from A.
From B:

$$\frac{1}{\tau}\int Q^{2k+2}\Lambda(\partial_\alpha^k(\tilde{\omega}))\frac{Q^2\tilde{\omega}^2}{|\tilde{z}_\alpha|}\phi_\delta * \phi_\delta * H(\partial_\alpha^k(\tilde{K})) \quad (B^1)$$

From C:

$$\frac{1}{|\tilde{z}_\alpha|\tau}\int Q^{2k+4}\tilde{\omega}^2\partial_\alpha^k(\tilde{\omega})\phi_\delta * \phi_\delta * \partial_\alpha^{k+1}(\tilde{K}) \quad (C^1)$$

We note that throughout this section we have repeatedly used the following commutator estimate for convolutions:

$$\|\phi_\delta * (\partial_\alpha f g) - g\phi_\delta * (\partial_\alpha f)\|_{L^2} \leq C\|\partial_\alpha g\|_{L^\infty}\|f\|_{L^2}, \tag{3.11}$$

where the constant C is independent of δ, f and g.

Also using this commutator estimate we can find all the cancelations we need in the previous collection of terms of low order type I and II to obtain a suitable energy estimate.

Regarding the high order terms, we will do the estimates in detail. We will see the need for the dissipative term since there are terms that escape for half of a derivative.

$$\begin{aligned}
A^2 + B^{2,2,1} + D &= 2\int Q^{2k+3}\left(\partial_\alpha^k(\tilde{K})\right)\phi_\delta * \phi_\delta * \left(H(\partial_\alpha^{k+2}\tilde{\omega})\right) \\
&\quad - 2\int Q^{2k+3}H(\partial_\alpha^{k+2}(\tilde{\omega}))\phi_\delta * \phi_\delta * \partial_\alpha^k(\tilde{K}) - 2\varepsilon\|\partial_\alpha^{k+1}\tilde{\omega}\|_{L^2}^2 \\
&= 2\int \partial_\alpha^k(\tilde{K})\left(Q^{2k+3}\phi_\delta * \phi_\delta * H(\partial_\alpha^{k+2}\tilde{\omega}) - \phi_\delta * \phi_\delta * \left(Q^{2k+3}H(\partial_\alpha^{k+2}\tilde{\omega})\right)\right) - 2\varepsilon\|\partial_\alpha^{k+1}\tilde{\omega}\|_{L^2}^2 \\
&\leq \|\partial_\alpha^k\tilde{K}\|_{L^2}\|\partial_\alpha Q^{2k+3}\|_{L^\infty}\|\partial_\alpha^{k+1}\tilde{\omega}\|_{L^2} - 2\varepsilon\|\partial_\alpha^{k+1}\tilde{\omega}\|_{L^2}^2 \leq C(\varepsilon)E^p(t),
\end{aligned}$$

which is uniform in δ. This proves that we can pass to the limit $\delta \to 0$.

Finally, by applying the a priori energy estimates to the new system (which only depend on ε) we can pass to the limit $\varepsilon \to 0$ since now we don't have the previous problems and $A^2 + B^{2,2,1} = 0$.

3.5 Energy with the Rayleigh-Taylor condition

In this section, we prove local existence in the tilde domain, where the time of existence does not depend on the surface tension coefficient. In this theorem, we need initial data to satisfy the Rayleigh-Taylor condition as we explain in Section 3.3. This Rayleigh-Taylor condition will hold in particular if the surface tension coefficient is small enough.

Theorem 3.5.1 *Let $k \geq 3$. Let $\tilde{z}^0(\alpha)$ be the image of a splash curve by the map P parametrized in such a way that $|\partial_\alpha \tilde{z}^0(\alpha)| = \frac{L}{2\pi}$, where L is the length of the curve in a fundamental period, and such that $\tilde{z}_1^0(\alpha), \tilde{z}_2^0(\alpha) \in H^{k+2}(\mathbb{T})$. Let $\tilde{\varphi}(\alpha, 0) \in H^{k+\frac{1}{2}}(\mathbb{T})$ be as in (3.3) and let $\tilde{\omega}(\alpha, 0) \in H^{k-1}(\mathbb{T})$. Then there exist a finite time $T > 0$, a time-varying curve $\tilde{z}(\alpha, t) \in C([0, T]; H^{k+2})$, and functions $\tilde{\omega}(\alpha, t) \in C([0, T]; H^{k-1})$ and $\tilde{\varphi} \in C([0, T]; H^{k+\frac{1}{2}})$ providing a solution of the water wave equations (3.1 - 3.2). Assume that initially, the Rayleigh-Taylor condition is strictly positive.*

In order to prove this theorem we will use the solutions we have obtained in theorem 3.4.1 for $\tau > 0$. We will perform energy estimates on these solutions.

3.5.1 The energy

We will define the energy for $k \geq 3$ as

$$
\begin{aligned}
E_k^2(t) = \mathcal{E}\mathcal{E}^2(t) &+ \underbrace{\tau \frac{|\tilde{z}_\alpha|}{2} \int Q^{2k+1} \left(\partial_\alpha^k (\tilde{K})\right)^2}_{A} + \underbrace{\int Q^{2k-2} \partial_\alpha^k (\tilde{\varphi}) \Lambda(\partial_\alpha^k (\tilde{\varphi}))}_{B} \\
&+ \underbrace{|\tilde{z}_\alpha|^2 \tau \int (\mathcal{C}\|\tilde{K}(t)\|_{H^1} + \tilde{K}) Q^{2k+1} \partial_\alpha^{k-1}(\tilde{K}) \Lambda(\partial_\alpha^{k-1}(\tilde{K}))}_{C} + \underbrace{2|\tilde{z}_\alpha| \int \mathcal{C}\|\tilde{K}(t)\|_{H^1} Q^{2k-2} \left(\partial_\alpha^k (\tilde{\varphi})\right)^2}_{D} \\
&+ \underbrace{|\tilde{z}_\alpha|^2 \int \sigma Q^{2k} \left(\partial_\alpha^{k-1}(\tilde{K})\right)^2}_{E} + \frac{|\tilde{z}_\alpha|^2}{m(Q^{2k}\sigma)(t)},
\end{aligned}
$$

where $m(Q^{2k}\sigma) = \min_{\alpha \in \mathbb{T}} Q^{2k}(\tilde{z}(\alpha, t))\sigma(\alpha, t)$ and $\mathcal{C}$ is a sufficiently large constant such that C is strictly positive. Remember that $\tilde{\varphi}$ was introduced in Equation 3.3.

At this point is important to notice the following.

Lemma 3.5.2 *The following sentences hold.*

1. *Let $\tilde{\varphi} \in H^{3+\frac{1}{2}}$, $\tilde{\omega} \in H^2$ and $z \in H^k$ with $k \geq 4$. Then $\tilde{\omega} \in H^3$.*
2. *Let $\tilde{\varphi} \in H^{3+\frac{1}{2}}$, $\tilde{\omega} \in H^3$ and $z \in H^k$ with $k \geq 5$. Then $\tilde{\omega} \in H^{3.5}$.*
3. *Let $\tilde{\omega} \in H^{3+\frac{1}{2}}$, and $\tilde{z} \in H^k$ with $k \geq 5$. Then $\tilde{\varphi} \in H^{3.5}$.*

This lemma shows that for a fixed $\tau > 0$ the energy of this section is equivalent to this one in section 3.4.1. This allows us to use this energy to extend the solutions of the theorem 3.4.1 up to a time T which does not depend on τ (for a small enough τ).

3.5.2 The energy estimates

Again, we will only focus on the new terms $(A - E)$ since the estimates for the other ones were proved in [28] and in [20].

3.5.2.1 $\tilde{K}$

Proposition 3.5.3

$$\begin{aligned}\tilde{K}_t &= \mathit{NICE3B} \ + \frac{Q^2}{2|\tilde{z}_\alpha|^3}H(\tilde{\omega}_{\alpha\alpha}) + \frac{1}{|\tilde{z}_\alpha|^3}(Q^2)_\alpha H(\tilde{\omega}_\alpha)\\ &= \mathit{NICE3B} \ + \frac{1}{|\tilde{z}_\alpha|^2}H(\tilde{\varphi}_{\alpha\alpha}) - \frac{1}{|\tilde{z}_\alpha|}(\tilde{K}\tilde{\varphi})_\alpha,\end{aligned}$$

where NICE3B means

$$\int Q^j \partial_\alpha^k(\tilde{K})\partial_\alpha^k(NICE3B) \leq CE_k^p(t)$$

for some positive constants C, p and any j.

Proof: The first equality follows from the proof from the last section since the energies are equivalent (see Lemma 3.5.2). We now prove the second one. We begin by using the relation (3.3) to get

$$\begin{aligned}\tilde{K}_t = \text{NICE3B} \ &+ \frac{Q^2}{|\tilde{z}_\alpha|^2}H\left(\left(\frac{\tilde{\varphi}}{Q^2}\right)_{\alpha\alpha}\right) + \frac{Q^2}{|\tilde{z}_\alpha|}H\left(\left(\frac{\tilde{c}}{Q^2}\right)_{\alpha\alpha}\right)\\ &+ \frac{2(Q^2)_\alpha}{|\tilde{z}_\alpha|^2}H\left(\left(\frac{\tilde{\varphi}}{Q^2}\right)_{\alpha}\right) + \frac{2(Q^2)_\alpha}{|\tilde{z}_\alpha|}H\left(\left(\frac{\tilde{c}}{Q^2}\right)_{\alpha}\right) = I + J\end{aligned}$$

We can easily see that

$$\tilde{c}_\alpha = -\frac{\tilde{z}_\alpha}{|\tilde{z}_\alpha|^2}\cdot(Q^2BR)_\alpha = \text{NICE3B}$$

since it is at the level of $\tilde{\omega}_\alpha, \tilde{z}_{\alpha\alpha}$ but we gain one derivative by multiplying by the tangential direction. This proves that

$$J = \text{NICE3B} \ + \frac{2(Q^2)_\alpha}{|\tilde{z}_\alpha|^2}H\left(\frac{\tilde{\varphi}_\alpha}{Q^2}\right).$$

Looking now to $\tilde{c}_{\alpha\alpha}$ we can see that

$$\tilde{c}_{\alpha\alpha} = -\frac{\tilde{z}_{\alpha\alpha}}{|\tilde{z}_\alpha|^2}\cdot(Q^2BR)_\alpha - \frac{\tilde{z}_\alpha}{|\tilde{z}_\alpha|^2}\cdot(Q^2BR)_{\alpha\alpha} = I_1 + I_2.$$

Using the standard estimates, the only thing that causes trouble in I_1 is when all the derivatives hit $\tilde{\omega}$ and therefore

$$I_1 = \text{NICE3B} \ - K\frac{Q^2}{2|\tilde{z}_\alpha|}H(\tilde{\omega}_\alpha).$$

Regarding I_2, again, we need all the derivatives to hit BR to get the most singular terms, which are

$$
\begin{aligned}
I_2 = {} & \text{NICE3B } - \frac{Q^2}{|\tilde{z}_\alpha|^2}\tilde{z}_\alpha \cdot \left[\frac{2}{2\pi}\int_{-\pi}^{\pi} \frac{(\tilde{z}_\alpha(\alpha) - \tilde{z}_\alpha(\beta))^\perp}{|\tilde{z}(\alpha) - \tilde{z}(\beta)|^2}\tilde{\omega}_\alpha(\alpha-\beta)d\beta\right] \\
& - \frac{Q^2}{|\tilde{z}_\alpha|^2}\tilde{z}_\alpha \cdot \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(\tilde{z}(\alpha) - \tilde{z}(\beta))^\perp}{|\tilde{z}(\alpha) - \tilde{z}(\beta)|^2}\tilde{\omega}_{\alpha\alpha}(\alpha-\beta)d\beta\right] \\
& - \frac{Q^2}{|\tilde{z}_\alpha|^2}\tilde{z}_\alpha \cdot \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(\tilde{z}_{\alpha\alpha}(\alpha) - \tilde{z}_{\alpha\alpha}(\beta))^\perp}{|\tilde{z}(\alpha) - \tilde{z}(\beta)|^2}\tilde{\omega}_\alpha(\alpha-\beta)d\beta\right] \\
& = \text{NICE3B } + \frac{2Q^2}{|\tilde{z}_\alpha|^2}\frac{1}{2}\frac{\tilde{z}_\alpha \cdot \tilde{z}_{\alpha\alpha}^\perp}{|\tilde{z}_\alpha|^2}H(\tilde{\omega}_\alpha) - \frac{Q^2}{|\tilde{z}_\alpha|^2}\frac{\tilde{z}_\alpha \cdot \tilde{z}_{\alpha\alpha}^\perp}{|\tilde{z}_\alpha|^2}H(\tilde{\omega}_\alpha) - \frac{Q^2}{|\tilde{z}_\alpha|^2}\frac{1}{2}\frac{\tilde{\omega}}{|\tilde{z}_\alpha|^2}\tilde{z}_\alpha \cdot H(\tilde{z}_{\alpha\alpha\alpha}^\perp)
\end{aligned}
$$

Collecting all the terms from I_1 and I_2, we obtain

$$
\begin{aligned}
\frac{\tilde{c}_{\alpha\alpha}}{Q^2} &= \text{NICE3B } + \frac{1}{2|\tilde{z}_\alpha|}H((\tilde{K}\tilde{\omega})_\alpha) \\
&= \text{NICE3B } + \frac{1}{Q^2}H((\tilde{K}\tilde{\varphi})_\alpha).
\end{aligned}
$$

We can finally write the total contribution as

$$
\begin{aligned}
\tilde{K}_t = {} & \text{NICE3B } + \frac{Q^2}{|\tilde{z}_\alpha|^2}H\left(\frac{\tilde{\varphi}_{\alpha\alpha}}{Q^2}\right) - \frac{Q^2}{|\tilde{z}_\alpha|^2}H\left(\frac{4Q_\alpha\tilde{\varphi}_\alpha}{Q^3}\right) \\
& - \frac{1}{|\tilde{z}_\alpha|}(\tilde{K}\tilde{\varphi})_\alpha + \frac{2(Q^2)_\alpha}{|\tilde{z}_\alpha|^2}H\left(\frac{\tilde{\varphi}_\alpha}{Q^2}\right) \\
& = \text{NICE3B } + \frac{Q^2}{|\tilde{z}_\alpha|^2}H\left(\frac{\tilde{\varphi}_{\alpha\alpha}}{Q^2}\right) - \frac{1}{|\tilde{z}_\alpha|}(\tilde{K}\tilde{\varphi})_\alpha \\
& = \text{NICE3B } + \frac{1}{|\tilde{z}_\alpha|^2}H\left(\tilde{\varphi}_{\alpha\alpha}\right) - \frac{1}{|\tilde{z}_\alpha|}(\tilde{K}\tilde{\varphi})_\alpha
\end{aligned}
$$

as we wanted to prove.

□

3.5.2.2 $\tilde{\varphi}$

Throughout this section, we will use the following estimate which was proved in [20] for the case without surface tension. The proof is exactly the same for the case with it.

$$
\varphi_{\alpha t} = \text{NICE2B } + \frac{\tilde{\varphi}\tilde{\varphi}_{\alpha\alpha}}{|\tilde{z}_\alpha|} - Q^2\sigma\tilde{K} + \tau\frac{Q^2}{2|\tilde{z}_\alpha|}\left((\tilde{K}Q)_\alpha + M_\alpha\right),
$$

where NICE2B means

$$
\int Q^j\Lambda(\partial_\alpha^k(\tilde{\varphi}))\partial_\alpha^{k-1}(NICE2B) \leq CE_k^p(t)
$$

for some positive constants C, p and any j.

3.5.3 Calculations of the time derivative of the energy

Using the previous lemmas and propositions, we can get the following estimates for the derivative of the energy:

$$\begin{aligned}\frac{dA}{dt} &= \text{OK } + \frac{\tau}{|\tilde{z}_\alpha|}\int Q^{2k+1}\partial_\alpha^k(\tilde{K})\partial_\alpha^k(H(\tilde{\varphi}_{\alpha\alpha})) - \tau\int Q^{2k+1}\partial_\alpha^k(\tilde{K})\partial_\alpha^k((\tilde{K}\tilde{\varphi})_\alpha)\\ &= \text{OK } + \frac{\tau}{|\tilde{z}_\alpha|}\int Q^{2k+1}\partial_\alpha^k(\tilde{K})\partial_\alpha^k(H(\tilde{\varphi}_{\alpha\alpha})) - \tau\int Q^{2k+1}\partial_\alpha^k(\tilde{K})\partial_\alpha^{k+1}(\tilde{\varphi})\tilde{K} = \text{OK } + A^1 + A^2\end{aligned}$$

Again, we need to be careful while computing the derivative of B as in Section 3.4. We obtain

$$\begin{aligned}\frac{dB}{dt} &= 2\int Q^{2k-2}\Lambda(\partial_\alpha^k(\tilde{\varphi}))\partial_\alpha^{k-1}(\tilde{\varphi}_{\alpha t}) + \int (Q^{2k-2})_\alpha H(\partial_\alpha^k(\tilde{\varphi}))\partial_\alpha^{k-1}(\tilde{\varphi}_{\alpha t})\\ &= \text{OK } - 2\int Q^{2k-2}\Lambda(\partial_\alpha^k(\tilde{\varphi}))\partial_\alpha^{k-1}\left(\frac{\tilde{\varphi}\tilde{\varphi}_{\alpha\alpha}}{|\tilde{z}_\alpha|}\right)\\ &- 2\int Q^{2k-2}\Lambda(\partial_\alpha^k(\tilde{\varphi}))\partial_\alpha^{k-1}(Q^2\sigma\tilde{K})\\ &+ \frac{\tau}{|\tilde{z}_\alpha|}\int Q^{2k-2}\Lambda(\partial_\alpha^k(\tilde{\varphi}))\partial_\alpha^k(Q^2(Q\tilde{K})_\alpha)\\ &- \frac{\tau}{|\tilde{z}_\alpha|}\int Q^{2k}Q_\alpha\Lambda(\partial_\alpha^k(\tilde{\varphi}))\partial_\alpha^k(\tilde{K})\\ &+ \int\frac{\tau}{|\tilde{z}_\alpha|}(k-1)Q^{2k+2}Q_\alpha H(\partial_\alpha^k(\tilde{\varphi}))\partial_\alpha^{k+1}(\tilde{K})\\ &= \text{OK } + B^1 + B^2 + B^3 + B^4 + B^5\end{aligned}$$

3.5.4 Development of the derivative of the B term

We begin noticing that $B^1 = \text{OK}$, as it was proved in [28]. Integrating by parts in B^5, we have that

$$B^5 = -\int\frac{\tau}{|\tilde{z}_\alpha|}(k-1)Q^{2k+2}Q_\alpha H(\partial_\alpha^{k+1}(\tilde{\varphi}))\partial_\alpha^k(\tilde{K})$$

Furthermore, the only singular terms arising from B^2 are when all derivatives hit either $\tilde{K}$ or σ, this gives us

$$B^2 = \text{OK } - 2\int Q^{2k}\Lambda(\partial_\alpha^k(\tilde{\varphi}))\partial_\alpha^{k-1}(\sigma)\tilde{K} - 2\int Q^{2k}\Lambda(\partial_\alpha^k(\tilde{\varphi}))\partial_\alpha^{k-1}(\tilde{K})\sigma = \text{OK } + B^{2,1} + B^{2,2}.$$

However, the only singular term of the Rayleigh-Taylor condition that is not in H^{k-1} is the one belonging to $BR_t(\tilde{z},\tilde{\omega})\cdot\tilde{z}_\alpha$ when the time derivative hits ω, this means

$$\begin{aligned}B^{2,1} &= \text{OK } - \tau\int Q^{2k}\Lambda(\partial_\alpha^k(\tilde{\varphi}))\tilde{K}H(\partial_\alpha^k(\tilde{K}Q))\\ &= -\tau\int Q^{2k+1}\Lambda(\partial_\alpha^k(\tilde{\varphi}))\tilde{K}H(\partial_\alpha^k(\tilde{K}))\end{aligned}$$

Finally, developing B^3 we obtain

$$\begin{aligned} B^3 &= \frac{\tau}{|\tilde{z}_\alpha|}\int Q^{2k-2}\Lambda(\partial_\alpha^k(\tilde{\varphi}))\partial_\alpha^k(Q^3\tilde{K}_\alpha) \\ &+ \frac{\tau}{|\tilde{z}_\alpha|}\int Q^{2k-2}\Lambda(\partial_\alpha^k(\tilde{\varphi}))\partial_\alpha^k(Q^2 Q_\alpha \tilde{K}) \\ &= B^{3,1} + B^{3,2} \end{aligned}$$

Modulo lower order terms we can see that

$$B^{3,2} = \text{OK } + \frac{\tau}{|\tilde{z}_\alpha|}\int Q^{2k}Q_\alpha\Lambda(\partial_\alpha^k(\tilde{\varphi}))\partial_\alpha^k(\tilde{K})$$

We can continue splitting $B^{3,1}$ into

$$\begin{aligned} B^{3,1} &= \text{OK } + \frac{\tau}{|\tilde{z}_\alpha|}\int Q^{2k+1}\Lambda(\partial_\alpha^k(\tilde{\varphi}))\partial_\alpha^{k+1}(\tilde{K}) + \frac{\tau}{|\tilde{z}_\alpha|}\int 3kQ^{2k}Q_\alpha\Lambda(\partial_\alpha^k(\tilde{\varphi}))\partial_\alpha^k(\tilde{K}) \\ &= \text{OK } - \frac{\tau}{|\tilde{z}_\alpha|}\int Q^{2k+1}\Lambda(\partial_\alpha^{k+1}(\tilde{\varphi}))\partial_\alpha^k(\tilde{K}) + \frac{\tau}{|\tilde{z}_\alpha|}\int (k-1)Q^{2k}Q_\alpha\Lambda(\partial_\alpha^k(\tilde{\varphi}))\partial_\alpha^k(\tilde{K}) \\ &= \text{OK } + B^{3,1,1} + B^{3,1,2} \end{aligned}$$

where in the last equality we have performed an integration by parts. We can observe that

$$B^{3,2} + B^4 = B^{3,1,2} + B^5 = 0, \quad B^{3,1,1} + A^1 = 0$$

We will now see that $B^{2,2}$ cancels with the term arising from the derivative of E. Taking into account the previous lemmas

$$\frac{dE}{dt} = 2\int \sigma Q^{2k}\left(\partial_\alpha^{k-1}(\tilde{K})\right)H(\partial_\alpha^{k+1}(\tilde{\varphi})) = \text{OK } - B^{2,2}$$

Finally, we will see that the contributions from the time derivatives of C and D cancel $B^{2,1}$ and A^2. We start by noticing that, modulo lower order terms $A^2 = B^{2,1}$. Furthermore

$$\begin{aligned} \frac{dC}{dt} &= \text{OK } + 2\tau\int (\mathcal{C}\|\tilde{K}(t)\|_{H^1} + \tilde{K})Q^{2k+1}H(\partial_\alpha^{k+1}(\tilde{\varphi}))\Lambda(\partial_\alpha^{k-1}(\tilde{K})) \\ \frac{dD}{dt} &= \text{OK } + 2\tau\int \mathcal{C}\|\tilde{K}(t)\|_{H^1}Q^{2k+1}\left(\partial_\alpha^k(\tilde{\varphi})\right)\partial_\alpha^{k+1}(\tilde{K}), \end{aligned}$$

which, by integration by parts results in

$$\frac{dC}{dt} + \frac{dD}{dt} + A^2 + B^{2,1} = \text{OK}.$$

Adding all the contributions, we can bound the derivative in time of the energy by a power of the energy.

3.6 Helpful estimates for the Birkhoff-Rott operator

In this Section we will prove some of the estimates used throughout the paper for the sake of clarity to the reader.

We begin with a classical decomposition of the Birkhoff-Rott operator. We should notice that we can write it in the following ways. On one hand:

$$
\begin{aligned}
BR(\tilde{z},\tilde{\omega})(\alpha) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{(\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta))^{\perp}}{|\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta)|^2}-\frac{\partial_\alpha^{\perp}\tilde{z}(\alpha)}{2|\partial_\alpha\tilde{z}(\alpha)|^2\tan(\beta/2)}\right)\tilde{\omega}(\alpha-\beta)d\beta \\
&+\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\partial_\alpha^{\perp}\tilde{z}(\alpha)}{2|\partial_\alpha\tilde{z}(\alpha)|^2\tan(\beta/2)}\right)\tilde{\omega}(\alpha-\beta)d\beta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{(\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta))^{\perp}}{|\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta)|^2}-\frac{\partial_\alpha^{\perp}\tilde{z}(\alpha)}{2|\partial_\alpha\tilde{z}(\alpha)|^2\tan(\beta/2)}\right)\tilde{\omega}(\alpha-\beta)d\beta+\frac{\partial_\alpha^{\perp}\tilde{z}(\alpha)}{2|\partial_\alpha\tilde{z}(\alpha)|^2}H(\tilde{\omega})(\alpha) \\
&= \frac{\partial_\alpha^{\perp}\tilde{z}(\alpha)}{2|\partial_\alpha\tilde{z}(\alpha)|^2}H(\tilde{\omega})(\alpha)+\text{l.o.t}(\tilde{\omega}).
\end{aligned}
$$

On the other hand:

$$
\begin{aligned}
BR(\tilde{z},\tilde{\omega})(\alpha) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta))^{\perp}}{|\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta)|^2}(\tilde{\omega}(\alpha-\beta)-\tilde{\omega}(\alpha))d\beta+\frac{\tilde{\omega}(\alpha)}{2\pi}\int_{-\pi}^{\pi}\frac{(\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta))^{\perp}}{|\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta)|^2}d\beta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta))^{\perp}}{|\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta)|^2}(\tilde{\omega}(\alpha-\beta)-\tilde{\omega}(\alpha))d\beta \\
&+\frac{\tilde{\omega}(\alpha)}{2\pi}\int_{-\pi}^{\pi}(\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta))^{\perp}\left(\frac{1}{|\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta)|^2}-\frac{1}{4|\tilde{z}_\alpha(\alpha)|^2\sin^2\left(\frac{\beta}{2}\right)}\right)d\beta \\
&+\frac{\tilde{\omega}(\alpha)}{2|\tilde{z}_\alpha(\alpha)|^2}\Lambda(\tilde{z}^{\perp}(\alpha)) \\
&= \frac{\tilde{\omega}(\alpha)}{2|\tilde{z}_\alpha(\alpha)|^2}\Lambda(\tilde{z}^{\perp}(\alpha))+\text{l.o.t}(\tilde{z}).
\end{aligned}
$$

See [8], [28] for more details concerning the lower order terms.

We will now prove energy estimates for the Birkhoff-Rott integral, showing that it is as regular as $\partial_\alpha\tilde{z}$. The proof is taken from [28, Section 6].

Lemma 3.6.1 *The following estimate holds*

$$
\|BR(\tilde{z},\tilde{\omega})\|_{H^k}\leq C(\|\mathcal{F}(\tilde{z})\|_{L^\infty}^2+\|\tilde{z}\|_{H^{k+1}}^2+\|\tilde{\omega}\|_{H^k}^2)^j, \tag{3.12}
$$

for $k\geq 2$, where C and j are constants independent of $\tilde{z}$ and $\tilde{\omega}$.

Remark 3.6.2 *Using this estimate for $k=2$ we find easily that*

$$
\|\partial_\alpha BR(\tilde{z},\tilde{\omega})\|_{L^\infty}\leq C(\|\mathcal{F}(\tilde{z})\|_{L^\infty}^2+\|\tilde{z}\|_{H^3}^2+\|\tilde{\omega}\|_{H^2}^2)^j. \tag{3.13}
$$

Proof: We shall present the proof for $k = 2$. Let us write

$$BR(\tilde{z}, \tilde{\omega})(\alpha, t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} C_1(\alpha, \beta)\tilde{\omega}(\alpha - \beta) d\beta + \frac{\partial_\alpha^\perp \tilde{z}(\alpha)}{2|\partial_\alpha \tilde{z}(\alpha)|^2} H(\tilde{\omega})(\alpha)$$

where C_1 is given by

$$C_1(\alpha, \beta) = \frac{(\tilde{z}(\alpha) - \tilde{z}(\alpha - \beta))^\perp}{|\tilde{z}(\alpha) - \tilde{z}(\alpha - \beta)|^2} - \frac{\partial_\alpha^\perp \tilde{z}(\alpha)}{2|\partial_\alpha \tilde{z}(\alpha)|^2 \tan(\beta/2)}, \tag{3.14}$$

We shall show that $\|C_1\|_{L^\infty} \leq C\|\mathcal{F}(\tilde{z})\|_{L^\infty}^2 \|\tilde{z}\|_{C^2}^2$. To do so we split $C_1 = D_1 + D_2 + D_3$ where

$$D_1 = \frac{(\tilde{z}(\alpha) - \tilde{z}(\alpha - \beta) - \partial_\alpha \tilde{z}(\alpha)\beta)^\perp}{|\tilde{z}(\alpha) - \tilde{z}(\alpha - \beta)|^2}, \quad D_2 = \partial_\alpha^\perp \tilde{z}(\alpha)[\frac{\beta}{|\tilde{z}(\alpha) - \tilde{z}(\alpha - \beta)|^2} - \frac{1}{|\partial_\alpha \tilde{z}(\alpha)|^2 \beta}],$$

and

$$D_3 = \frac{\partial_\alpha^\perp \tilde{z}(\alpha)}{|\partial_\alpha \tilde{z}(\alpha)|^2}[\frac{1}{\beta} - \frac{1}{2\tan(\beta/2)}].$$

The inequality

$$|\tilde{z}(\alpha) - \tilde{z}(\alpha - \beta) - \partial_\alpha \tilde{z}(\alpha)\beta| \leq \|\tilde{z}\|_{C^2} |\beta|^2 \tag{3.15}$$

yields easily $|D_1| \leq \|\tilde{z}\|_{C^2} \|\mathcal{F}(\tilde{z})\|_{L^\infty}^2$.

Then we can rewrite D_2 as follows:

$$D_2 = \partial_\alpha^\perp \tilde{z}(\alpha)[\frac{(\partial_\alpha \tilde{z}(\alpha)\beta - (\tilde{z}(\alpha) - \tilde{z}(\alpha - \beta))) \cdot (\partial_\alpha \tilde{z}(\alpha)\beta + (\tilde{z}(\alpha) - \tilde{z}(\alpha - \beta)))}{|\tilde{z}(\alpha) - \tilde{z}(\alpha - \beta)|^2 |\partial_\alpha \tilde{z}(\alpha)|^2 \beta}],$$

and, in particular, we have

$$|D_2| \leq \frac{|\partial_\alpha \tilde{z}(\alpha)\beta - (\tilde{z}(\alpha) - \tilde{z}(\alpha - \beta))|(|\partial_\alpha \tilde{z}(\alpha)\beta| + |\tilde{z}(\alpha) - \tilde{z}(\alpha - \beta)|)}{|\tilde{z}(\alpha) - \tilde{z}(\alpha - \beta)|^2 |\partial_\alpha \tilde{z}(\alpha)||\beta|}.$$

Using (3.15) we find that $|D_2| \leq 2\|\tilde{z}\|_{C^2} \|\mathcal{F}(\tilde{z})\|_{L^\infty}^2$.

Next let us observe that since $\beta \in [-\pi, \pi]$ gives $|D_3| \leq C\|\mathcal{F}(\tilde{z})\|_{L^\infty}$.

The boundedness of the term C_1 in L^∞ gives us easily

$$\|BR(\tilde{z}, \tilde{\omega})\|_{L^2} \leq C\|\mathcal{F}(\tilde{z})\|_{L^\infty}^2 \|\tilde{z}\|_{C^2}^2 \|\tilde{\omega}\|_{L^2}. \tag{3.16}$$

In $\partial_\alpha^2 BR(\tilde{z}, \tilde{\omega})$, the most singular terms are given by

$$P_1(\alpha) = \frac{1}{2\pi} PV \int_{-\pi}^{\pi} \partial_\alpha^2 \tilde{\omega}(\alpha - \beta) \frac{(\tilde{z}(\alpha) - \tilde{z}(\alpha - \beta))^\perp}{|\tilde{z}(\alpha) - \tilde{z}(\alpha - \beta)|^2} d\beta,$$

$$P_2(\alpha) = \frac{1}{2\pi} PV \int_{-\pi}^{\pi} \tilde{\omega}(\alpha - \beta) \frac{(\partial_\alpha^2 \tilde{z}(\alpha) - \partial_\alpha^2 \tilde{z}(\alpha - \beta))^\perp}{|\tilde{z}(\alpha) - \tilde{z}(\alpha - \beta)|^2} d\beta,$$

$$P_3(\alpha) = -\frac{1}{\pi} PV \int_{-\pi}^{\pi} \tilde{\omega}(\alpha - \beta) \frac{(\tilde{z}(\alpha) - \tilde{z}(\alpha - \beta))^\perp}{|\tilde{z}(\alpha) - \tilde{z}(\alpha - \beta)|^4} \big(\tilde{z}(\alpha) - \tilde{z}(\alpha - \beta)\big) \cdot \big(\partial_\alpha^2 \tilde{z}(\alpha) - \partial_\alpha^2 \tilde{z}(\alpha - \beta)\big) d\beta.$$

Again we have the expression

$$P_1(\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi} C_1(\alpha,\beta)\partial_\alpha^2\tilde{\omega}(\alpha-\beta)d\beta + \frac{\partial_\alpha^\perp\tilde{z}(\alpha)}{2|\partial_\alpha\tilde{z}(\alpha)|^2}H(\partial_\alpha^2\tilde{\omega})(\alpha)d\alpha,$$

giving us

$$|P_1(\alpha)| \leq C\|\mathcal{F}(\tilde{z})\|_{L^\infty}^j\|\tilde{z}\|_{C^2}^j(\|\partial_\alpha^2\tilde{\omega}\|_{L^2} + |H(\partial_\alpha^2\tilde{\omega})(\alpha)|). \tag{3.17}$$

Next let us write $P_2 = Q_1 + Q_2 + Q_3$ where

$$Q_1(\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\tilde{\omega}(\alpha-\beta)-\tilde{\omega}(\alpha))\frac{(\partial_\alpha^2\tilde{z}(\alpha)-\partial_\alpha^2\tilde{z}(\alpha-\beta))^\perp}{|\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta)|^2}d\beta,$$

$$Q_2(\alpha) = \frac{\tilde{\omega}(\alpha)}{2\pi}\int_{-\pi}^{\pi}(\partial_\alpha^2\tilde{z}(\alpha)-\partial_\alpha^2\tilde{z}(\alpha-\beta))^\perp\Big(\frac{1}{|\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta)|^2} - \frac{1}{|\partial_\alpha\tilde{z}(\alpha)|^2|\beta|^2}\Big)d\beta,$$

$$Q_3(\alpha) = \frac{1}{2\pi}\frac{\tilde{\omega}(\alpha)}{|\partial_\alpha\tilde{z}(\alpha)|^2}\int_{-\pi}^{\pi}(\partial_\alpha^2\tilde{z}(\alpha)-\partial_\alpha^2\tilde{z}(\alpha-\beta))^\perp\Big(\frac{1}{|\beta|^2}-\frac{1}{4\sin^2(\beta/2)}\Big)d\beta+\frac{1}{2}\frac{\tilde{\omega}(\alpha)}{|\partial_\alpha\tilde{z}(\alpha)|^2}\Lambda(\partial_\alpha^2\tilde{z}^\perp)(\alpha),$$

where $\Lambda = \partial_\alpha H$.

Using that

$$|\partial_\alpha^2\tilde{z}(\alpha)-\partial_\alpha^2\tilde{z}(\alpha-\beta)| \leq |\beta|^\delta\|\tilde{z}\|_{C^{2,\delta}},$$

we get $|Q_1(\alpha)|+|Q_2(\alpha)| \leq \|\tilde{\omega}\|_{C^1}\|\mathcal{F}(\tilde{z})\|^j\|\tilde{z}\|_{C^{2,\delta}}^j$, while for Q_3 we have

$$|Q_3(\alpha)| \leq C\|\tilde{\omega}\|_{L^\infty}\|\mathcal{F}(\tilde{z})\|_{L^\infty}(\|\tilde{z}\|_{C^2} + |\Lambda(\partial_\alpha^2\tilde{z}^\perp)(\alpha)|),$$

that is

$$|P_2(\alpha)| \leq (1+|\Lambda(\partial_\alpha^2\tilde{z}^\perp)(\alpha)|)\|\tilde{\omega}\|_{C^1}\|\mathcal{F}(\tilde{z})\|^j\|\tilde{z}\|_{C^{2,\delta}}^j. \tag{3.18}$$

Let us now consider $P_3 = Q_4+Q_5+Q_6+Q_7+Q_8+Q_9$, where

$$Q_4 = \frac{-1}{\pi}\int_{-\pi}^{\pi}(\tilde{\omega}(\alpha-\beta)-\tilde{\omega}(\alpha))\frac{(\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta))^\perp}{|\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta)|^4}\big((\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta))\cdot(\partial_\alpha^2\tilde{z}(\alpha)-\partial_\alpha^2\tilde{z}(\alpha-\beta))\big)d\beta,$$

$$Q_5 = -\frac{\tilde{\omega}(\alpha)}{\pi}\int_{-\pi}^{\pi}\frac{(\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta)-\partial_\alpha\tilde{z}(\alpha)\beta)^\perp}{|\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta)|^4}\big((\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta))\cdot(\partial_\alpha^2\tilde{z}(\alpha)-\partial_\alpha^2\tilde{z}(\alpha-\beta))\big)d\beta,$$

$$Q_6 = -\frac{\tilde{\omega}(\alpha)\partial_\alpha^\perp\tilde{z}(\alpha)}{\pi}\int_{-\pi}^{\pi}\frac{\beta(\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta)-\partial_\alpha\tilde{z}(\alpha)\beta)\cdot(\partial_\alpha^2\tilde{z}(\alpha)-\partial_\alpha^2\tilde{z}(\alpha-\beta))}{|\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta)|^4}d\beta,$$

$$Q_7 = -\frac{\tilde{\omega}(\alpha)\partial_\alpha^\perp\tilde{z}(\alpha)}{\pi}\partial_\alpha\tilde{z}(\alpha)\cdot\int_{-\pi}^{\pi}\beta^2(\partial_\alpha^2\tilde{z}(\alpha)-\partial_\alpha^2\tilde{z}(\alpha-\beta))\Big(\frac{1}{|\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta)|^4}-\frac{1}{|\partial_\alpha\tilde{z}(\alpha)|^4|\beta|^4}\Big)d\beta,$$

$$Q_8 = -\frac{\tilde{\omega}(\alpha)\partial_\alpha^\perp\tilde{z}(\alpha)}{\pi|\partial_\alpha\tilde{z}(\alpha)|^4}\partial_\alpha\tilde{z}(\alpha)\cdot\int_{-\pi}^{\pi}(\partial_\alpha^2\tilde{z}(\alpha)-\partial_\alpha^2\tilde{z}(\alpha-\beta))\Big(\frac{1}{|\beta|^2}-\frac{1}{4\sin^2(\beta/2)}\Big)d\beta,$$

and

$$Q_9 = -\frac{\tilde{\omega}(\alpha)\partial_\alpha^\perp\tilde{z}(\alpha)}{|\partial_\alpha\tilde{z}(\alpha)|^4}\partial_\alpha\tilde{z}(\alpha)\cdot\Lambda(\partial_\alpha^2\tilde{z}(\alpha)).$$

Proceeding as before we get

$$|P_3(\alpha)| \leq C(1+|\Lambda(\partial_\alpha^2\tilde{z})(\alpha)|)\|\tilde{\omega}\|_{C^1}\|\mathcal{F}(\tilde{z})\|^j_{L^\infty}\|\tilde{z}\|^j_{C^{2,\delta}},$$

which together with (3.17) and (3.18) gives us the estimate

$$|(P_1+P_2+P_3)(\alpha)| \leq C(1+|\Lambda(\partial_\alpha^2\tilde{z})(\alpha)|+|H(\partial_\alpha^2\tilde{\omega})(\alpha)|)\|\tilde{\omega}\|_{C^1}(\|\mathcal{F}(\tilde{z})\|^j_{L^\infty}+\|\tilde{z}\|^j_{H^3}).$$

For the rest of the terms in $\partial_\alpha^2 BR(\tilde{z},\tilde{\omega})$ we obtain analogous estimates allowing us to conclude the equality

$$\|\partial_\alpha^2 BR(\tilde{z},\tilde{\omega})\|_{L^2} \leq C(1+\|\partial_\alpha^3\tilde{z}\|_{L^2}+\|\partial_\alpha^2\tilde{\omega}\|_{L^2})\|\tilde{\omega}\|_{C^1}\|\mathcal{F}(\tilde{z})\|^j_{L^\infty}\|\tilde{z}\|^j_{C^{2,\delta}}.$$

Finally the Sobolev inequalities yield (3.20) for $k=2$. □

Lemma 3.6.3 *The following estimate will also be helpful*

$$\|\partial_\alpha BR(\tilde{z},\tilde{\omega})\cdot\partial_\alpha\tilde{z}\|_{H^k} \leq C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty}+\|\tilde{z}\|^2_{H^{k+2}}+\|\tilde{\omega}\|^2_{H^k})^j, \tag{3.19}$$

for $k\geq 2$, where C and j are constants independent of $\tilde{z}$ and $\tilde{\omega}$.

Proof: In $\partial_\alpha BR(\tilde{z},\tilde{\omega})\cdot\partial_\alpha\tilde{z}$, the most singular terms are given by

$$R_1(\alpha)=\frac{1}{2\pi}PV\int_{-\pi}^{\pi}\partial_\alpha\tilde{\omega}(\alpha-\beta)\frac{\partial_\alpha\tilde{z}(\alpha)\cdot(\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta))^\perp}{|\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta)|^2}d\beta,$$

$$R_2(\alpha)=\frac{1}{2\pi}PV\int_{-\pi}^{\pi}\tilde{\omega}(\alpha-\beta)\frac{\partial_\alpha\tilde{z}(\alpha)\cdot(\partial_\alpha\tilde{z}(\alpha)-\partial_\alpha\tilde{z}(\alpha-\beta))^\perp}{|\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta)|^2}d\beta,$$

$$R_3(\alpha)=-\frac{1}{\pi}PV\int_{-\pi}^{\pi}\tilde{\omega}(\alpha-\beta)\frac{\partial_\alpha\tilde{z}(\alpha)\cdot(\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta))^\perp}{|\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta)|^4}\big(\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta)\big)\cdot\big(\partial_\alpha\tilde{z}(\alpha)-\partial_\alpha\tilde{z}(\alpha-\beta)\big)d\beta.$$

R_2 can be estimated in the same way as P_2. Regarding R_1, one can write it as

$$R_1(\alpha)=\frac{1}{2\pi}PV\int_{-\pi}^{\pi}\partial_\alpha\tilde{\omega}(\alpha-\beta)\left[\frac{\partial_\alpha\tilde{z}(\alpha)\cdot(\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta))^\perp}{|\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta)|^2}-\frac{\partial_\alpha\tilde{z}_\alpha\cdot\partial_\alpha\tilde{z}_\alpha^\perp}{|\tilde{z}_\alpha(\alpha)|^2}\right]d\beta.$$

Now, since $\partial_\alpha\tilde{\omega}(\alpha-\beta)=-\partial_\beta\tilde{\omega}(\alpha-\beta)$, one can integrate by parts and bound the resulting kernel (which has order -1) giving

$$|R_1|\leq C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty}+\|\tilde{z}\|^2_{H^{k+2}}+\|\tilde{\omega}\|^2_{H^k})^j.$$

Finally, R_3 can be written in the form

$$R_3(\alpha)=-\frac{1}{\pi}PV\int_{-\pi}^{\pi}\tilde{\omega}(\alpha-\beta)\left[\frac{\partial_\alpha\tilde{z}(\alpha)\cdot(\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta))^\perp}{|\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta)|^4}\big(\tilde{z}(\alpha)-\tilde{z}(\alpha-\beta)\big)\cdot\big(\partial_\alpha\tilde{z}(\alpha)-\partial_\alpha\tilde{z}(\alpha-\beta)\big)\right.$$
$$\left.-\frac{\partial_\alpha\tilde{z}(\alpha)\cdot\partial_\alpha\tilde{z}(\alpha)^\perp}{\beta|\tilde{z}_\alpha(\alpha)|^4}\partial_\alpha\tilde{z}(\alpha)\cdot\partial_\alpha^2\tilde{z}(\alpha)\right]d\beta,$$

and bound R_3 by the kernel (which has order 0) in L^∞ norm and ω in L^2 norm. This completes the proof.

□

Then, the following corollary is immediate

Corollary 3.6.4

$$\|\tilde{c}_\alpha\|_{H^k} \leq C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty} + \|\tilde{z}\|^2_{H^{k+2}} + \|\tilde{\omega}\|^2_{H^k})^j, \tag{3.20}$$

for $k \geq 2$, where C and j are constants independent of $\tilde{z}$ and $\tilde{\omega}$.

Chapter 4

Introduction to Computer-Assisted Proofs

4.1 Computer-Assisted Proofs and Interval arithmetics

In the last 50 years computing power has experienced an enormous development. According to Moore's Law [72], every two years the number of transistors has doubled since the 1970's. This phenomenon has resulted in the blooming of new techniques located in the verge between pure mathematics and computational ones. However, even nowadays when we can perform computations at the speeds of the order of Petaflops (a quatrillion floating point operations per second) we can not avoid the following questions, still fundamental in the rigorous analysis of the output of a computer program:

Q1: Is a computer result influenced by the way the individual operations are done?

Q2: Does the environment (operating system, computer architecture, compiler, rounding modes, ...) have any impact on the result?

Sadly, the answer to these questions is Yes, which can be easily illustrated by the following C++ codes (see Listings 4.1 and 4.3). The first one computes the harmonic series up to a given N in two ways: the first way adds the different numbers from the bigger ones to the smaller and the second one does the sum in the opposite way. The results for $N = 10^6$ can be seen in Listing 4.2. They are not the same and curiously, the real result is not any of the two of them. The second program uses the MPFR library [47] to add two numbers given by the user in two different ways: rounding down and rounding up the result. The output is done in binary. We can see that the results differ (Listing 4.4).

Listing 4.1: Computation of the truncated Harmonic Series in two different ways

```
int main(int argc, char* argv[]){
  int N; cin >> N;
  cout.setf(ios::fixed); cout.precision(15);
  double res1, res2; res1 = res2 = 0.0;
  for (int i=1; i<=N; i++){
```

```
    res1 = res1 + 1.0/(double)i;
  }
  for (int i=N; i>=1; i--){
    res2 = res2 + 1.0/(double)i;
  }
  cout << res1 << endl;
  cout << res2 << endl;
}
```

Listing 4.2: Result of the previous computation.

```
14.3927267228647811
14.3927267228657563
```

Listing 4.3: Sum of two numbers with different rounding

```
int main (int argc, char **argv){
  mpfr_t x, y, d, u;
  mpfr_prec_t prec;
  prec = atoi (argv[1]);
  int pprec = prec - 1;
  mpfr_inits2 (prec, x, y, d, u, (mpfr_ptr) 0);
  mpfr_set_str (x, argv[2], 0, GMP_RNDN);
  mpfr_printf ("x = %.*Rb\n", pprec, x);
  mpfr_set_str (y, argv[3], 0, GMP_RNDN);
  mpfr_printf ("y = %.*Rb\n", pprec, y);

  mpfr_add (d, x, y, GMP_RNDD);
  mpfr_printf ("d = %.*Rb\n", pprec, d);

  mpfr_add (u, x, y, GMP_RNDU);
  mpfr_printf ("u = %.*Rb\n", pprec, u);

  return 0;
}
```

Listing 4.4: Program executed with arguments 10 0.1 1

```
x = 1.100110011p-4
y = 1.000000000p+0
d = 1.000110011p+0
u = 1.000110100p+0
```

This shows that even the simplest algorithms need a careful analysis: only two operations suffice to give different results if executed in different order or with different rounding methods.

Fortunately, the theory of interval analysis developed by R. Moore [73] is an example of a tool, which albeit being impractical due to inefficient resources at the time of its conception, is now being widely used. It belongs to the paradigm known as rigorous computing (in some contexts also called validated computing), in which numerical computations are used to provide rigorous mathematical statements about a result. The philosophy behind the theory of interval analysis consists in working with and producing objects which are not numbers, but intervals in which we are sure that the true result lies. Therefore the answer to the second question is also Yes. Nevertheless, we should be precise enough since even with plenty of resources, overestimation might lead to too big intervals which might not guarantee the desired result.

Lately, interval methods have become quite popular among mathematicians. Several highly non-trivial results have been established by the use of interval arithmetics, see for example [58, 48, 81] as a small sample. However, there is a fraction of the mathematical community for which there exist doubts whether one can rely on the fundamentals of the physics (in the sense that computers do the right thing according to given physical laws) to prove a rigorous mathematical theorem or not [1]. Even so, it is my belief that after having seen the successful outcomes, it is clear that the future of mathematics must include techniques for performing validated numerics.

In analysis, the most celebrated result is the proof of the dynamics of the Lorenz attractor (Smale's 14th Problem) done by Tucker [87] in 2002. However, the study of the dynamics of a system has been restricted almost always to (typically low-dimensional) ODEs. As an example, we can cite the following papers related to the N-body problem, Rössler equations and the Henon map [66, 67, 93], but there is a big literature on the topic. Another problems involving ODEs but an infinite dimensional system are the computation of the ground state energy of atoms or the relativistic stability of matter (see [82, 42, 41]).

Regarding PDEs (infinite dimension problems), most of the work has been carried out for dissipative systems (i.e. systems in which the L^2-norm of the function decreases with time). The most popular ones are the Kuramoto-Sivashinsky equations or Navier-Stokes in low dimensions (typically 1). The main feature of these models is that one can study the first N modes of the Fourier expansion of the function and see the rest as an "error". Since the system is dissipative, if N is large enough, one can get a control on the error throughout time. We should remark here that the linearization of the water waves equation shows that the system is not dissipative, but dispersive. Other techniques (such as the proof of existence of periodic orbits, for example) reduce the problem to compute the norm of an adequate operator between Banach spaces and apply Brouwer's fixed point theorem to show that the operator has a fixed point (the orbit) or compute the Conley index of a certain region. These methods have been applied for instance in [94] (Conley index for Kuramoto-Sivashinsky), [10] (Bifurcation diagram for stationary solutions of Kuramoto-Sivashinsky), [37] (Global atractors for viscous 1D Burgers), [45] (Stationary solutions of viscous 1D Burgers with boundary conditions), [44] (Traveling wave solutions for 1D Burgers equation), [43] (Newton scheme around an approximate solution following the spirit of the one described in the next section, for Kuramoto-Sivashinsky).

Representing an abstract concept such as a real number by a finite number of zeros and ones has the advantage that the calculations are finite and the framework is practical. The drawback is naturally that the amount of numbers that can be written in this way is finite (although of the same order of magnitude as the age of the universe in seconds) and inaccuracies might arise while performing mathematical operations. We will now discuss the basics of interval arithmetics. From now on, unless otherwise stated, we will suppose that the numbers are represented using 64 bits. All the reasoning can be easily reinterpreted in the case of arbitrary high precision (multiprecision).

Let $\mathbb{F}$ be the set of representable numbers by a computer. We will work with the set of representable closed intervals $\mathbb{IR} = \{[\underline{a}, \overline{a}] | \quad \underline{a} \leq \overline{a}, \quad \underline{a}, \overline{a} \in \mathbb{F}\}$. For every element $[a] \in \mathbb{IR}$ we will refer to it by either $[a]$ or by $[\underline{a}, \overline{a}]$, whenever we want to stress the importance of the endpoints of the interval. We can now define an arithmetic by the theoretic-set definition

$$[x] \star [y] = \{x \star y | \quad x \in [x], y \in [y]\}, \tag{4.1}$$

for any operation $\star \in \{+, -, \times, \div\}$. We can easily define them by the following equations:

$$\begin{aligned}
[x] + [y] &= [\underline{x} + \underline{y}, \overline{x} + \overline{y}] \\
[x] - [y] &= [\underline{x} + \overline{y}, \overline{x} + \underline{y}] \\
[x] \times [y] &= [\min\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}, \max\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}] \\
[x] \div [y] &= [x] \times \left[\frac{1}{\overline{y}}, \frac{1}{\underline{y}}\right], \text{ whenever } 0 \notin [y].
\end{aligned}$$

Note that this interval-valued operators can be extended to other algebraic expressions involving exponential, trigonometric, inverse trigonometric functions, etc. This derivation is purely theoretical, and we should keep in mind that, if carried out on a computer, the results of an operation have to be rounded up or down according to whether we are calculating the left or right endpoint so that the true result is enclosed in the produced interval. The main feature of the arithmetic is that if $x \in [x], y \in [y]$, then necessarily $x \star y \in [x] \star [y]$ for any operator $\star$. This property is fundamental in order to ensure that the true result is always contained in the interval we get from the computer.

We remark that this arithmetic is not distributive, but subdistributive, i.e:

$$\begin{aligned}
[a] \times ([b] + [c]) &\neq [a] \times [b] + [a] \times [c] \\
[a] \times ([b] + [c]) &\subset [a] \times [b] + [a] \times [c]
\end{aligned}$$

Example 4.1.1 *If we set* $[a] = [3, 4]$, $[b] = [1, 2]$, $[c] = [-1, 1]$, *then:*

$$\begin{aligned}
[a] \times ([b] + [c]) &= [0, 12] \\
[a] \times [b] + [a] \times [c] &= [-1, 12]
\end{aligned}$$

This illustrates that the way in which operations are executed in the interval-based arithmetic matters much more than in the real-based. As an example, consider the function $f(x) = 1 - x^2$ and a domain $D = [-1, 1]$. Over the reals, we can write f as any of the following functions:

$$\begin{aligned}
f_1(x) &= 1 - x^2 \\
f_2(x) &= 1 - x \cdot x \\
f_3(x) &= (1 + x) \cdot (1 - x)
\end{aligned}$$

However, evaluating f_i over D we get the enclosures:

$$\begin{aligned} f_1(D) &= [0,1] \\ f_2(D) &= [0,2] \\ f_3(D) &= [0,4] \end{aligned}$$

We observe that although f_3 is completely factored, if we expand it we get an expression of the form $x - x$ which in the interval-based arithmetic is equal to an interval of a width twice the width of the domain in which we are evaluating the expression: a price too high to pay compared with the width of the interval $[0,0]$, another form to write the same expression over the reals.

4.2 Automatic Differentiation

One of the main tasks in which we will need the help of a computer is to calculate a massive amount of function evaluations and their derivatives up to a given order at several points and intervals. In order to perform it, one could first think of trying to differentiate the expressions symbolically. However, we don't need the expression of the derivative, just its evaluation at given points. This, together with the fact that the size of the derivative might grow exponentially, makes the use of symbolic calculus impractical. Instead of calculating the expression of every derivative, we will use the so-called *automatic differentiation* methods. Suppose $f(x)$ is a sufficiently regular function and let x_0 be the point (or interval) of which we want to calculate its image by f. We define

$$\begin{aligned} (f)_0 &= f(x_0) \\ (f)_k &= \frac{1}{k!}\frac{d^k}{dx^k} f(x_0), \quad k = 1, 2, \ldots, N, \end{aligned}$$

where N stands for the maximum number of derivatives of the function we want to evaluate. We can think about (f) as being the coefficients of the Taylor series around x_0 up to order N. We now show how to compute the coefficients (f) for some of the functions that will appear in our programs. The generalization of the missing functions is immediate. However, it is possible to derive similar formulas for any solution of a differential equation (e.g. Bessel functions).

$$\begin{aligned} (u \pm v)_k &= (u)_k \pm (v)_k \\ (u \cdot v)_k &= \sum_{j=0}^{k} (u)_j (v)_{k-j} \\ (u \div v)_k &= (1/v) \left((u)_k - \sum_{j=1}^{k} (v)_j (u \div v)_{k-j} \right) \\ (\sin(u))_k &= \frac{1}{k} \sum_{j=0}^{k-1} (j+1)(\cos(u))_{k-1-j} (u)_{j+1} \end{aligned}$$

$$(\cos(u))_k = -\frac{1}{k}\sum_{j=0}^{k-1}(j+1)(\sin(u))_{k-1-j}(u)_{j+1}$$

Automatic differentiation has become a natural technique in the field of Dynamical Systems, since the cost for evaluating an expression up to order k is $O(k^2)$, making it a fast and powerful tool to approximate accurately trajectories [84]. It has also been used for the computation of invariant tori and their associated invariant manifolds [57, 56] or the computation of normal forms of KAM tori [54]. For more applications in Dynamical Systems we refer the reader to the survey [55]. Automatic Differentiation is also an important element in the so-called Taylor models [76, 71, 64], in which functions are represented by couples (P, Δ), being P a polynomial and Δ an interval bound on the absolute value of the difference between the function and P. Nowadays, there are several packages that implement it, for example [65, 12].

4.3 Integration

In this section we will discuss the basics of rigorous integration. A more detailed version concerning singular integrals of piecewise-defined functions can be found in the next Chapter. We will only give the details of the one-dimensional case, omitting the multidimensional one. A brief description of a two-dimensional rigorous integration method can be found in the last chapter.

The main problem we address here is to calculate bounds for a given integral

$$I = \int_a^b f(x)dx, \quad -\infty < a < b < \infty.$$

Different strategies can be used for this purpose. For instance, we can extend the classical integration schemes:

$$I = \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} f(x)dx, \quad a = x_0 < x_1 < \ldots < x_N = b.$$

In every interval, we approximate $f(x)$ by a polynomial $p(x)$ and an error term. We detail some typical examples in Table 4.1: It is now clear where the interval arithmetic takes place. In order to enclose the value of the integral, we need to compute rigorous bounds for some derivative of the function at the integration region.

Another approach consists of taking the Taylor series of the integrand up to order n as the polynomial $p_i(x)$. Centering the Taylor series in the midpoint of the interval makes us integrate only roughly over half of the terms (since the other half are equal to zero). We can see that

	Midpoint Rule	Trapezoid Rule	Simpson's Rule
	$I \approx \sum_{i=1}^{N} p_i(x)dx$		
$p_i(x)$	$f\left(\frac{x_i+x_{i-1}}{2}\right)$	$f(x_{i-1})\left(1-\frac{x-x_{i-1}}{h_i}\right) + f(x_i)\left(\frac{x-x_{i-1}}{h_i}\right)$	$f(x_{i-1})\frac{(x-x_i)(x-\frac{x_i+x_{i-1}}{2})}{h_i^2/2} - f\left(\frac{x_i+x_{i-1}}{2}\right)\frac{(x-x_i)(x-x_{i-1})}{h_i^2/4} + f(x_i)\frac{(x-x_{i-1})(x-\frac{x_i+x_{i-1}}{2})}{h_i^2/2}$
Error	$\frac{b-a}{24}h^2f^2([a,b])$	$-\frac{b-a}{12}h^2f^2([a,b])$	$-\frac{b-a}{2880}h^4f^4([a,b])$

Table 4.1: Different Schemes for the rigorous integration.

$$
\begin{aligned}
\int_a^b f(x)dx &= \int_a^b f(a) + (x-a)f'(a) + \ldots + \frac{(x-a)^n}{n!}f^n(a) + \frac{(x-a)^{n+1}}{(n+1)!}f^{n+1}(\xi(x))dx \\
&\in \int_a^b f(a) + (x-a)f'(a) + \ldots + \frac{(x-a)^n}{n!}f^n(a) + \frac{(x-a)^{n+1}}{(n+1)!}f^{n+1}([a,b])dx \\
&= \underbrace{(b-a)f(a) + \frac{1}{2}(b-a)^2f'(a) + \ldots + \frac{(x-a)^{n+1}}{(n+1)!}f^n(a)}_{\text{Real number (thin interval)}} + \underbrace{\frac{(x-a)^{n+2}}{(n+2)!}f^{n+1}([a,b])}_{\text{Error (thick interval)}}.
\end{aligned}
$$

We now compare the two methods in the following examples, in which we integrate $\int_0^1 e^x dx$.

Example 4.3.1 *If we take $N = 4$ and use a trapezoidal rule, we enclose the integral in*

$$
\begin{aligned}
\int_0^1 e^x dx &= \frac{1}{2}\left(e^0 + 2e^{1/4} + 2e^{1/2} + 2e^{3/4} + e^1\right)\frac{1}{4} - \frac{1}{12}\frac{1}{16}e^{[0,1]} \\
&\in [1.72722, 1.72723] - [0.0050283, 0.014578]
\end{aligned}
$$

Example 4.3.2

$$
\begin{aligned}
\int_0^1 e^x dx &\in \int_0^1 1 + x + \frac{x^2}{2} + \frac{x^3}{6}e^{[0,1]}dx \\
&= x + \frac{x^2}{2} + \frac{x^3}{6}\Big|_{x=0}^{x=1} + [1,e]\frac{x^4}{24}\Big|_{x=0}^{x=1} \\
&= \frac{10}{6} + \frac{1}{24}[1,e] \\
&= [1.70833, 1.77994]
\end{aligned}
$$

The exact result is $e - 1 \approx 1.71828182846$. We can see that there is a tradeoff between function evaluations (efficiency of the scheme) and quality (precision) of the results, since the first method is more exact but requires more evaluations of the integrand, while for the second it is enough to compute the Taylor series of the integrand.

Chapter 5

From a graph to a Splash Singularity

5.1 Introduction

In this Chapter we will give details on a possible proof of the following result:

Conjecture 5.1.1 *There exist initial data $z_0(\alpha), \omega_0(\alpha)$ that are solutions of the water wave equations such that at time 0 they can be parametrized as a graph, then turn over at a finite time $T_1 > 0$, and finally produce a splash at a finite time $T_2 > T_1$.*

We should remark that this conjecture is a combination of Theorem 2.1.1 and [22, Theorem 7.1] and is supported by numerical evidence. The proof follows along this lines. First of all, we will move backwards in time, being 0 the time of the splash, $T_2 - T_1$ the time of the turning and T_2 the time in which the solution can be parametrized as a graph. We start computing a numerical approximation of a solution (x, γ, ψ) to the water waves equation that starts as a splash, turns over and finally is a graph. Such a candidate is depicted in Figures 2.1,3.1. Another ingredient is the following stability result that was announced in [19], that will allow us to conclude the following: if (x, γ) approximately satisfies equation (5.1), then near to (x, γ) there exists an exact solution (z, ω). Below is the theorem.

Theorem 5.1.2 *Let*

$$D(\alpha, t) \equiv z(\alpha, t) - x(\alpha, t), \quad d(\alpha, t) \equiv \omega(\alpha, t) - \gamma(\alpha, t), \quad \mathcal{D}(\alpha, t) \equiv \varphi(\alpha, t) - \psi(\alpha, t)$$

where (x, γ, ψ) *are the solutions of*

$$\begin{cases} x_t &= Q^2(x)BR(x,\gamma) + bx_\alpha + f \\ b &= \underbrace{\frac{\alpha+\pi}{2\pi}\int_{-\pi}^{\pi}(Q^2BR(x,\gamma))_\alpha \frac{x_\alpha}{|x_\alpha|^2}d\alpha - \int_{-\pi}^{\alpha}(Q^2BR(x,\gamma))_\beta \frac{x_\alpha}{|x_\alpha|^2}d\beta}_{b_s} \\ &+ \underbrace{\frac{\alpha+\pi}{2\pi}\int_{-\pi}^{\pi} f_\alpha \frac{x_\alpha}{|x_\alpha|^2}d\alpha - \int_{-\pi}^{\alpha} f_\beta \frac{x_\beta}{|x_\beta|^2}d\beta}_{b_e} \\ \gamma_t & +2BR_t(x,\gamma)\cdot x_\alpha = -(Q^2(x))_\alpha |BR(x,\gamma)|^2 + 2bBR_\alpha(x,\gamma)\cdot x_\alpha + (b\gamma)_\alpha \\ & \qquad - \left(\frac{Q^2(x)\gamma^2}{4|x_\alpha|^2}\right)_\alpha - 2(P_2^{-1}(x))_\alpha + g \\ \psi(\alpha,t) &= \frac{Q_x^2(\alpha,t)\gamma(\alpha,t)}{2|x_\alpha(\alpha,t)|} - b_s(\alpha,t)|x_\alpha(\alpha,t)|, \end{cases} \tag{5.1}$$

(z, ω, φ) *are the solutions of* (5.1) *with* $f \equiv g \equiv 0$ *and* E *the following norm for the difference*

$$E(t) \equiv \left(\|D\|_{H^3}^2 + \int_{-\pi}^{\pi} \frac{Q^2\sigma_z}{|z_\alpha|^2}|\partial_\alpha^4 D|^2 + \|d\|_{H^2}^2 + \|\mathcal{D}\|_{H^{3+\frac{1}{2}}}^2 \right).$$

Then we have that

$$\left|\frac{d}{dt}E(t)\right| \leq \mathcal{C}(t)(E(t) + E^k(t)) + c\delta(t)$$

where

$$\mathcal{C}(t) = \mathcal{C}(\mathcal{E}(t), \|x\|_{H^{5+\frac{1}{2}}}(t), \|\gamma\|_{H^{3+\frac{1}{2}}}(t), \|\zeta\|_{H^{4+\frac{1}{2}}}(t), \|F(x)\|_{L^\infty}(t))$$

and

$$\delta(t) = (\|f\|_{H^{5+\frac{1}{2}}}(t), \|g\|_{H^{3+\frac{1}{2}}}(t))^k, \text{ } k \text{ big enough}$$

depend on the norms of f *and* g, *and* $\mathcal{E}(t)$ *is given by*

$$\begin{aligned} \mathcal{E}(t) =& \|z\|_{H^3}^2(t) + \int_{\mathbb{T}} \frac{Q^2\sigma_z}{|z_\alpha|^2}|\partial_\alpha^4 z|^2 d\alpha + \|F(z)\|_{L^\infty}^2(t) \\ &+ \|\omega\|_{H^2}^2(t) + \|\varphi\|_{H^{3+\frac{1}{2}}}^2(t) + \frac{|z_\alpha|^2}{m(Q^2\sigma_z)(t)} + \sum_{l=0}^{4}\frac{1}{m(q^l)(t)} \end{aligned}$$

where the L^∞ *norm of the function*

$$F(z) \equiv \frac{|\beta|}{|z(\alpha,t) - z(\alpha-\beta,t)|}, \quad \alpha,\beta \in \mathbb{T}$$

measures the arc-chord condition,

$$\begin{aligned} \sigma_z \equiv& \left(BR_t(z,\omega) + \frac{\psi}{|z_\alpha|}BR_\alpha(z,\omega)\right)\cdot z_\alpha^\perp + \frac{\omega}{2|z_\alpha|^2}\left(z_{\alpha t} + \frac{\psi}{|z_\alpha|}z_{\alpha\alpha}\right)\cdot z_\alpha^\perp \\ &+ Q\left|BR(z,\omega) + \frac{\omega}{2|z_\alpha|^2}z_\alpha\right|^2 (\nabla Q)(z)\cdot z_\alpha^\perp + (\nabla P_2^{-1})(z)\cdot z_\alpha^\perp \end{aligned} \tag{5.2}$$

is the Rayleigh-Taylor function,

$$m(Q^2\sigma_z)(t) \equiv \min_{\alpha\in\mathbb{T}} Q^2(\alpha,t)\sigma_z(\alpha,t),$$

and finally

$$m(q^l)(t) \equiv \min_{\alpha\in\mathbb{T}} |z(\alpha,t) - q^l|$$

for $l = 0, ..., 4$.

Remark 5.1.3 *We can absorb the terms in* $\mathcal{E}(t)$ *by* $E(t)$ *raised to an appropriate power and terms in* (x,γ,ψ) *by performing the splitting* $\|z\| = \|z-x\| + \|x\|$ *(or the analogous one for a different variable) for any norm or any quantity that appears in* $\mathcal{E}(t)$.

We can construct a solution (z,ω,φ) that satisfies (5.1) with $f = g = 0$ and very similar same initial conditions

$$z_0(\alpha) \approx x_0(\alpha), \quad \omega_0(\alpha) \approx \gamma_0(\alpha), \quad \varphi_0(\alpha) \approx \psi_0(\alpha).$$

If we knew $\mathcal{C}(t), f(t), g(t), k$ or bounds on them, a priori, then we could provide bounds on $\mathcal{E}(t)$ at any time T. We point out here that $E(t)$ controls the norm $\|\partial_\alpha z^1(\alpha) - \partial_\alpha x^1(\alpha)\|_{L^\infty}$. Let T_g be a time in which the approximate solution is a graph, i.e. $\partial_\alpha x^1(\alpha, T_g) > 0 \quad \forall\alpha$. Now, if $E(T_g) < \partial_\alpha x^1(\alpha, T_g)$ then

$$\partial_\alpha z^1(\alpha, T_g) > \|\partial_\alpha z^1(\alpha) - \partial_\alpha x^1(\alpha)\|_{L^\infty} - \partial_\alpha x^1(\alpha, T_g) > 0,$$

and this shows that z is a graph. In other words, the possible set of solutions of the water waves equation is a ball centered at (x,γ,ζ) with the topology given by E. All of the elements of this ball are graphs, therefore the solution is necessarily a graph. Thus, the problem is reduced to study and find bounds for $\mathcal{C}(t), f(t), g(t), k$.

5.2 Bounds for $f(t)$ and $g(t)$

5.2.1 Representation of the functions and Interpolation

The first thing one has to decide is how to represent the data and how to pass from the cloud of points in space-time obtained by non-rigorous simulation to a function defined everywhere in $[-\pi,\pi]\times[0,T]$. We need to interpolate in some way. One of the first things that can come to one's mind is to use the first N Fourier modes. This approach has two disadvantages. The first one is that the linearized water waves equation is not dissipative, hence we will not have a control for the tails uniformly for all time, even in the case where the tails have very small norms at time zero. The second disadvantage is numerical. Suppose $N \sim 10^3$, which is a reasonable guess. Since we need to take 5.5 derivatives in the curve, the high order coefficients will be multiplied by roughly a factor of 10^{15}. If we work with a 64-bit representation, machine epsilon is of the order 10^{-16} and we will run into trouble because the computer will not distinguish between zero and non-zero values. Of course, this problem can be solved if we use high precision arithmetics, but in any case we would be multiplying and dividing by very big constants.

In our case, we chose to represent the functions x and γ by piecewise polynomials (splines) of high degree (10) in space, and low degree (3) in time. To do so, we first interpolate in space for every node in the time mesh. The interpolation is made via B-Splines. Since the interpolation is reduced to solve a linear (interval) system $Ac = y$, where A is constant in time and space and y depends on the values of the function at time t since the mesh in space is constant, we precondition by multiplying by the non-rigorous inverse of the midpoints of the entries of A. We remark that the system is interval-based because we need to produce a curve that is a splash (i.e. there have to be two points α_1, α_2 such that we can guarantee $x_0(\alpha_1) = x_0(\alpha_2)$. Finally, the system is solved using a rigorous Gauss-Seidel iterative method. We also remark that the need for interval-based calculations is only strictly necessary at time $t = 0$ since it is the only point in which we have to guarantee some equality. By working with multiprecision (1024 bits) we can get widths in the coefficients of the order of 10^{-300}. In order to perform interpolation in time, we fix the values of the function and its time derivative at the mesh points. This gives us lots of systems of 4 equations (the values of the function and its derivative at both endpoints) and 4 unknowns (the 4 coefficients of the degree 3 polynomial) but with an explicit formula for each of them. With this method, our spline will be C^1 in time but it might not be C^2.

5.2.2 Rigorous bounds for Singular integrals

In this section we will discuss the computational details of the rigorous calculation of some singular integrals. In particular we will focus on the Hilbert transform, but the methods apply to any singular operator of order -1. Parts of the computation (the N part) are slightly related to the Taylor models with relative remainder presented in M. Joldeş' thesis [64].

Let us suppose that we have a function f given explicitly by a spline (piecewise polynomial) which is C^{k-1} everywhere and C^k everywhere but in finite points (the points in which the different pieces of the spline are glued together. We need to calculate rigorously the Hilbert Transform of f, that is

$$Hf(x) = \frac{PV}{\pi}\int_{\mathbb{T}} \frac{f(x)-f(y)}{2\tan\left(\frac{x-y}{2}\right)}dy,$$

and we want to approximate it by a piecewise polynomial function with less regularity, plus an error that can be bounded in $H^q, 0 \leq q \leq c < k$ and in L^∞. Let us assume that the knots of the spline are α_i, $i = 0, \ldots, N-1$ and that we fix $x \in [\alpha_i, \alpha_{i+1}]$ where the indices are taken modulo N and the distance between the indices is taken over $\mathbb{Z}_N$. We can split our integral in

$$
\begin{aligned}
Hf(x) &= \frac{PV}{\pi}\int_{\mathbb{T}} \frac{f(x)-f(y)}{2\tan\left(\frac{x-y}{2}\right)}dy = \frac{PV}{\pi}\sum_j \int_{\alpha_j}^{\alpha_{j+1}} \frac{f(x)-f(y)}{2\tan\left(\frac{x-y}{2}\right)}dy \\
&= \frac{PV}{\pi}\sum_{|j-i|>K} \int_{\alpha_j}^{\alpha_{j+1}} \frac{f(x)-f(y)}{2\tan\left(\frac{x-y}{2}\right)}dy + \frac{PV}{\pi}\sum_{|j-i|\leq K} \int_{\alpha_j}^{\alpha_{j+1}} \frac{f(x)-f(y)}{2\tan\left(\frac{x-y}{2}\right)}dy \\
&\equiv Hf^F(x) + Hf^N(x).
\end{aligned}
$$

Now, if we want to express $Hf^F(x)$ as a polynomial, it is easy since the integrand does not have a singularity. Hence

$$
\begin{aligned}
Hf^F(x) &= \frac{PV}{\pi}\sum_{|j-i|>K} \int_{\alpha_j}^{\alpha_{j+1}} \frac{f(x)-f(y)}{2\tan\left(\frac{x-y}{2}\right)}dy = \frac{PV}{\pi}\sum_{|j-i|>K} \int_{\alpha_j}^{\alpha_{j+1}} F^j(x,y)dy \\
&= \sum_{|j-i|>K} \int_{\alpha_j}^{\alpha_{j+1}} \sum_{n,m} c_{nm}(x-x^*(i))^m(y-y^*(j))^n + E(x,y)dy \equiv P(x) + E(x),
\end{aligned}
$$

where E accounts for the error and is a polynomial with interval coefficients. Typically, we will use as the points for the Taylor expansions $x^*(i) = \alpha_i$ since we will compare the resulting polynomial with another one of the form $\sum_j b_j(x-x_i)^j$ and we will also choose $y^*(j) = \frac{\alpha_j+\alpha_{j+1}}{2}$. This choice is twofold: first, we will only have to integrate half of the terms since the rest will integrate to zero; and second, the error estimates will be better for this choice of $y^*(j)$ in the sense that the coefficients will be smaller. All the computations will be carried out using automatic differentiation. We should remark that we can get estimates for the error E in any of the above mentioned norms without having to recompute it since the relation

$$\partial_x^q Hf^F(x) - \partial_x^q P(x) = \partial_x^q E(x)$$

holds for every $q < k$.

Now, we move on the the term $Hf^N(x)$. In this case, we perform a Taylor expansion in both the denominator

$$2\tan\left(\frac{x-y}{2}\right) = (x-y) + c(x-y)^3, \quad c = \text{ small (interval) constant}$$

and the numerator

$$f(x) = f(y) + (x-y)f'(y) + \frac{1}{2}(x-y)^2 f''(y) + \ldots \frac{1}{n!}(x-y)^{k-1}f^{k-1}(\eta),$$

where η belongs to an intermediate point between x and y, which we can enclose in the convex hull of $[\alpha_i, \alpha_{i+1}]$ and $[\alpha_j, \alpha_{j+1}]$ where the convex hull is understood in the torus. Since typically K will be very small (compared to N) there is no ambiguity in the definition. Finally, we can factor out $(x-y)$ and divide both in the numerator and the denominator. Since we know $f(y)$ explicitly, we can perform the explicit integration and get a piecewise polynomial as a result.

5.2.3 Estimates of the norm of the Operator $I + T$

In this subsection we will outline how to compute the norm of the operator $I + T = I + 2\langle BR(z, \cdot), z_\alpha \rangle$. Since the operator T behaves like a Hilbert Transform plus smoothing terms, we will describe how to calculate rigorously with the help of a computer an estimate for the norm of its inverse. The procedure is more general and can be applied to a bigger family of kernels. Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, and let $A(x), B(x)$ be real-valued functions on $\mathbb{T}$. Also, let $E(x, y)$ be a real-valued function on $\mathbb{T} \times \mathbb{T}$. We assume A, B and E are given by explicit formulas such as as perhaps piecewise trigonometric polynomials or splines, and $E(x, y)$ is a trigonometric polynomial on each rectangle $I \times J$ of some partition of $\mathbb{T} \times \mathbb{T}$. We suppose A, B, E are smooth enough.

Let H be the Hilbert transform acting on functions on $\mathbb{T}$, i.e.

$$Hf(x) = \frac{PV}{2\pi} \int_{\mathbb{T}} \cot\left(\frac{y}{2}\right) f(x - y) dy.$$

Assume that A and B have no common zeros on $\mathbb{T}$.

Let

$$Sf(x) = A(x)f(x) + B(x)Hf(x) + \int_{\mathbb{T}} E(x, y)f(y)dy, \quad f \in L^2(\mathbb{T}).$$

Thus, S is a singular integral operator.

We hope that S^{-1} exists and has a not-so-big norm on L^2, but we don't know this yet.

Our goal here is to find approximate solutions F of the equation $SF = f$ for suitable given $f \in L^2(\mathbb{T})$, and to check that $\|SF - f\|_{L^2(\mathbb{T})} < \delta$ for suitable δ. Our computation of F will be based on heuristic ideas, but the computation of an upper bound for $\|SF - f\|_{L^2(\mathbb{T})}$ will be rigorous. In our case, $A(x) = 1, B(x) = 1$.

To carry this out, let $H_0 \subset H_1 \subset L^2(\mathbb{T})$ be finite-dimensional subspaces, e.g. with H_i consisting of the span of wavelets (from a wavelet bases) having lengthscale $\geq 2^{-N_i}$. Here $N_1 \geq N_0 + 3$ (say). Let π_i be the orthogonal projection from $L^2(\mathbb{T})$ to H_i, and let us solve the equation

$$\pi_1 S \pi_1 F = \pi_0 f. \tag{5.3}$$

If f is given explicitly in a wavelet bases, then (5.3) is a linear algebra problem, since $\pi_1 S \pi_1$ is of finite rank, and its matrix (in terms of some given basis for H_1) can be computed explicitly.

- If $\pi_0 f \notin \text{Range}(\pi_1 S \pi_1)$, then our heuristic procedure fails.
- If $\pi_0 f \in \text{Range}(\pi_1 S \pi_1)$, then we find $F \in H_1$ such that $\pi_1 S \pi_1 F = \pi_0 f$, i.e. $\pi_1 S F = \pi_0 f$.

We then have

$$\|SF - f\|_{L^2(\mathbb{T})} \leq \|(I - \pi_1)SF\|_{L^2(\mathbb{T})} + \|(I - \pi_0)f\|_{L^2(\mathbb{T})},$$

and both norms on the right-hand side may be estimated explicitly.

Now, our goal is to make a heuristic computation of an operator of the form

$$\tilde{S}f(x) = \tilde{A}(x)f(f) + \tilde{B}(x)Hf(x) + \int_{\mathbb{T}} \tilde{E}(x,y)f(y)dy$$

such that $S\tilde{S} - I$ has small norm on $L^2(\mathbb{T})$.

Here, we will make a heuristic computation of $\tilde{S}$; later we will give a rigorous upper bound for the norm of $S\tilde{S} - I$ on $L^2(\mathbb{T})$. By a heuristic computation of $\tilde{S}$ we mean a heuristic computation of $\tilde{A}, \tilde{B}$ and $\tilde{E}$.

We first find $\tilde{A}$ and $\tilde{B}$ by setting

$$(A + iB)(\tilde{A} + i\tilde{B}) = 1 \Rightarrow \begin{cases} A\tilde{A} - B\tilde{B} & = 1 \\ A\tilde{B} + B\tilde{A} & = 0 \end{cases}$$

Then, this means that

$$S\tilde{S} = (A\tilde{A} - B\tilde{B}) + (A\tilde{B} + B\tilde{A})H + \text{ Smoothing terms} = I + \text{ Smoothing terms}$$

So, from now on, we suppose that $\tilde{A}$ and $\tilde{B}$ are known. For the operator $I+T$, this means $\tilde{A} = 1/2, \tilde{B} = -1/2$. We want to compute $\tilde{E}$. Now, let $\{\phi_\nu\}$ be some orthonormal basis for $L^2(\mathbb{T})$, for example a wavelet basis. By the previous methods, we can try to find functions $\psi_\nu \in L^2(\mathbb{T})$ such that $S\psi_\nu - \phi_\nu$ has small norm. We carry this for $\nu = 1, \ldots, N$ for a large N. We now try to make $\tilde{E}$ satisfy

$$\tilde{A}(x)\phi_\nu(x) + \tilde{B}(x)H\phi_\nu(x) + \int_{\mathbb{T}} \tilde{E}(x,y)\phi_\nu(y) = \psi_\nu(x) \text{ for } \nu = 1, \ldots, N. \tag{5.4}$$

Thus, we want

$$\int_{\mathbb{T}} \tilde{E}(x,y)\phi_\nu(y)dy = \Big(\psi_\nu(x) - \tilde{A}(x)\phi_\nu(x) - \tilde{B}(x)H\phi_\nu(x)\Big) \equiv \psi_\nu^{\#}(x), \quad \nu = 1, \ldots, N. \tag{5.5}$$

Note that $\psi_\nu^{\#}$ can be computed explicitly.

Since the ϕ_ν (all ν) form an orthonormal basis for $L^2(\mathbb{T})$, it is natural to define

$$\tilde{E}(x,y) = \sum_{\nu=1}^{N} \psi_\nu^{\#}\phi_\nu(y).$$

This can be computed explicitly, and it satisfies (5.5). Thus, we can the compute

$$\begin{aligned} S\tilde{S} &= (A + BH + E)(\tilde{A} + \tilde{B}H + \tilde{E}) \\ &= A\tilde{A} + A\tilde{B}H + A\tilde{E} + BH\tilde{A} + BH\tilde{B}H + BH\tilde{E} + E\tilde{A} + E\tilde{B}H + E\tilde{E} \\ &= A\tilde{A} + A\tilde{B}H + A\tilde{E} + B\tilde{A}H + B[H,\tilde{A}] - B\tilde{B} + B[H,\tilde{B}]H \\ &+ BH\tilde{E} + E\tilde{A} + E\tilde{B}H + E\tilde{E} \\ &= (A\tilde{A} - B\tilde{B}) + (A\tilde{B} + B\tilde{A})H + \{A\tilde{E} + B[H,\tilde{A}] + B[H,\tilde{B}]H \\ &+ BH\tilde{E} + E\tilde{A} + E\tilde{B}H + E\tilde{E}\} \end{aligned} \tag{5.6}$$

We claim that all terms enclosed in curly brackets are integral operators of the form

$$S^{\#}f(x) = \int_{\mathbb{T}} E^{\#}(x,y)f(y)dy,$$

for an $E^{\#}$ that we can calculate. Let us go term by term

- $A\tilde{E}$ has the form $S^{\#}$, with $E^{\#}(x,y) = A(x)\tilde{E}(x,y)$.
- $B[H,\tilde{A}]$ has the form $S^{\#}$, with $E^{\#}(x,y) = \frac{1}{2\pi}B(x)\cot\left(\frac{x-y}{2}\right)(\tilde{A}(x) - \tilde{A}(y))$.

  Note that if $\tilde{A}$ is a piecewise trigonometric polynomial and C^k, then $E^{\#}$ can easily be computed modulo a small error in C^{k-1}.
- $B[H,\tilde{B}]H$ has the form $S^{\#}$, with

$$\begin{aligned} E^{\#}(x,y) &= \frac{1}{4\pi^2}B(x)PV\int \cot\left(\frac{x-z}{2}\right)(\tilde{B}(x)-\tilde{B}(z))\cot\left(\frac{z-y}{2}\right)dz. \\ &= \frac{1}{4\pi^2}B(x)PV\int \left\{\cot\left(\frac{x-z}{2}\right)(\tilde{B}(x)-\tilde{B}(z)) - 2\tilde{B}'(x)\right\}\cot\left(\frac{z-y}{2}\right)dz. \end{aligned}$$

- $BH\tilde{E}$ has the form $S^{\#}$, with

$$\begin{aligned} E^{\#}(x,y) &= \frac{1}{2\pi}B(x)PV\int \cot\left(\frac{x-z}{2}\right)\tilde{E}(z,y)dz. \\ &= \frac{1}{2\pi}B(x)PV\int \cot\left(\frac{x-z}{2}\right)\left(\tilde{E}(z,y)-\tilde{E}(x,y)\right)dz. \end{aligned}$$

- $E\tilde{A}$ has the form $S^{\#}$, with $E^{\#}(x,y) = \tilde{E}(x,y)\tilde{A}(y)$.
- $E\tilde{B}H$ has the form $S^{\#}$, with

$$\begin{aligned} E^{\#}(x,y) &= \frac{1}{2\pi}PV\int E(x,z)\tilde{B}(z)\cot\left(\frac{z-y}{2}\right)dz. \\ &= \frac{1}{2\pi}PV\int \left\{E(x,z)\tilde{B}(z) - E(x,y)\tilde{B}(y)\right\}\cot\left(\frac{z-y}{2}\right)dz. \end{aligned}$$

- $E\tilde{E}$ has the form $S^{\#}$, with $E^{\#}(x,y) = \int E(x,z)\tilde{E}(z,y)dz$.

This proves the claim.

Letting $\mathcal{E}^{\#}f(x) = \int_{\mathbb{T}} E^{\#}(x,y)f(y)dy$ be the operator in curly brackets in (5.6), we see that

$$S\tilde{S} = (A\tilde{A} - B\tilde{B}) + (A\tilde{B} + B\tilde{A})H + \mathcal{E}^{\#},$$

and that the function $E^{\#}(x,y)$ can be computed modulo a small error in $C^0(\mathbb{T}\times\mathbb{T})$. Therefore, we obtain an upper boundfor the norm of $S\tilde{S} - I$, namely

$$\max|A\tilde{A} - B\tilde{B} - 1| + \max|A\tilde{B} + B\tilde{A}| + \max\left\{\max_x \int |E^{\#}(x,y)|dy, \max_y \int |E^{\#}(x,y)|dx\right\}.$$

Defining $S_{err} := S\tilde{S} - I$, we obtain an explicit upper bound δ for the norm of S_{err} on $L^2(\mathbb{T})$. We hope that $\delta < 1$. If not, then we fail.

Suppose $\delta < 1$. Then

$$S\tilde{S} = I + S_{err} \Rightarrow S\tilde{S}(I + S_{err})^{-1} = I,$$

so we obtain a right inverse for S, namely $\tilde{S}(I + S_{err})^{-1}$, which has norm at most

$$\|\tilde{S}\|(1-\delta)^{-1}, \tag{5.7}$$

where $\|\tilde{S}\|$ denotes the norm of $\tilde{S}$ as an operator on $L^2(\mathbb{T})$. Recall

$$\tilde{S}f(x) = \tilde{A}(x)f(x) + \tilde{B}(x)Hf(x) + \int_{\mathbb{T}} \tilde{E}(x,y)f(y)dy.$$

Therefore,

$$\|\tilde{S}\| \le \max|\tilde{A}(x)| + \max|\tilde{B}(x)| + \max\left\{\max_x \int |\tilde{E}(x,y)|dy, \max_y \int |\tilde{E}(x,y)|dx\right\}.$$

Plugging that bound into (5.7), we obtain an explicit upper bound for the norm on L^2 of a right inverse for S. Similarly (by looking at $\tilde{S}S$ instead of $S\tilde{S}$), we obtain an upper bound for the norm on L^2 of a left inverse for S.

Remark 5.2.1 *To estimate e.g.* $\max_x \int_{\mathbb{T}} |E^{\#}(x,y)|dy$ *it may be enough just to use the trivial estimate*

$$\max_x \int_{\mathbb{T}} |E^{\#}(x,y)|dy \le 2\pi \max_{x,y} |E^{\#}(x,y)|$$

Remark 5.2.2 (Time dependent solutions) *For* $t \in [t_0, t_1]$ *(a small time interval), let*

$$S_t f(x) = A(x,t)f(x) + B(x,t)Hf(x) + \int_{\mathbb{T}} E(x,y,t)f(y)dy,$$

where (for each t*),*$A(\cdot,t), B(\cdot,t), E(\cdot,\cdot,t)$ *are as assumed above.*

If A, B, E *depend in a reasonable way on* t*, then one shows easilly that*

$$\|S_t - S_{t_0}\| < \eta \text{ for all } t \in [t_0, t_1].$$

We can make η *small by taking* t_1 *close enough to* t_0*. Suppose we prove that* $\|S_{t_0}^{-1}\| \le C_0$ *by the previous methods. Then, of course we obtain an upper bound for* $\|S_t^{-1}\|$ *valid for all* $t \in [t_0, t_1]$*.*

5.3 Bounds for $\mathcal{C}(t)$ and k

5.3.1 Writing the differential inequality as a differential system of equations

The calculation of a bound for $\mathcal{C}(t)$ requires more effort than the previous one since one needs to calculate the terms one by one and add all their contributions to $\mathcal{C}(t)$. For example, in order to calculate the evolution of the norm $\|D\|_{H^k}(t)$ a systematic approach is to take k derivatives (k ranging from 0 to 4) in the equation for the evolution of z (5.1 with $f = g = 0$), take another k derivatives in the equation for x (5.1 with arbitrary f, g) and subtract them. Let us focus from now on in the term $Q(z)^2BR(z,\omega) - Q(x)^2BR(x,\gamma)$ and its derivatives. One notices that in order to write a term in the variables (z,ω,φ) composed of a factors minus its counterpart in the variables (x,γ,η) in a suitable way (i.e. as a sum of terms that only have factors $x,\gamma,\eta,D,d,\mathcal{D}$) then the number of terms is $2^a - 1$. The way of writing it is the classical way of adding and subtracting the same term with the purpose of creating differences of terms and eliminate all the occurrences of the variables (z,ω,φ). An example for the Birkhoff-Rott operator (with $Q = 1$) is given next. We should remark that the computation and bounding of the Birkhoff-Rott is the most expensive one, being easier the rest of the terms.

$$
\begin{aligned}
&BR(z,\omega) - BR(x,\gamma) = \frac{1}{2\pi}\int \frac{(x(\alpha)-x(\beta))^\perp}{|x(\alpha)-x(\beta)|^2}\,(\omega(\beta)-\gamma(\beta))\,d\beta\\
&+\frac{1}{2\pi}\int \frac{(z(\alpha)-z(\beta))^\perp-(x(\alpha)-x(\beta))^\perp}{|x(\alpha)-x(\beta)|^2}\,(\gamma(\beta))\,d\beta\\
&+\frac{1}{2\pi}\int \frac{(z(\alpha)-z(\beta))^\perp-(x(\alpha)-x(\beta))^\perp}{|x(\alpha)-x(\beta)|^2}\,(\omega(\beta)-\gamma(\beta))\,d\beta\\
&+\frac{1}{2\pi}\int \left(\frac{1}{|z(\alpha)-z(\beta)|^2}-\frac{1}{|x(\alpha)-x(\beta)|^2}\right)(x(\alpha)-x(\beta))^\perp\gamma(\beta)d\beta\\
&+\frac{1}{2\pi}\int \left(\frac{1}{|z(\alpha)-z(\beta)|^2}-\frac{1}{|x(\alpha)-x(\beta)|^2}\right)(x(\alpha)-x(\beta))^\perp(\omega(\beta)-\gamma(\beta))d\beta\\
&+\frac{1}{2\pi}\int \left(\frac{1}{|z(\alpha)-z(\beta)|^2}-\frac{1}{|x(\alpha)-x(\beta)|^2}\right)(z(\alpha)-z(\beta)-(x(\alpha)-x(\beta)))^\perp\gamma(\beta)d\beta\\
&+\frac{1}{2\pi}\int \left(\frac{1}{|z(\alpha)-z(\beta)|^2}-\frac{1}{|x(\alpha)-x(\beta)|^2}\right)(z(\alpha)-z(\beta)-(x(\alpha)-x(\beta)))^\perp\omega(\beta)-\gamma(\beta))d\beta
\end{aligned}
$$

After having seen this, it is clear that a tool that can perform symbolic calculations (derivation and basic arithmetic at least) and the correct grouping of the factors is required since the performance at this task by a human is not satisfactory. We developed a tool in 900 lines of C++ code that could do all this and output the collection of terms in Tex. We show an excerpt of the terms concerning the fourth derivative of $BR(z,\omega) - BR(x,\gamma)$. The total number of terms in that case is 2841.

$$
\begin{aligned}
&2\pi\left(\partial_\alpha^4 BR(x,\gamma)-\partial_\alpha^4 BR(z,\omega)\right)=\\
&+\int(\partial_\alpha^4 x(\alpha)-\partial_\alpha^4 x(\alpha-\beta))^\perp d(\alpha-\beta)\frac{1}{|x(\alpha)-x(\alpha-\beta)|^2}d\alpha\\
&+\int(\partial_\alpha^4 x(\alpha)-\partial_\alpha^4 x(\alpha-\beta))^\perp d(\alpha-\beta)\left(\frac{1}{|x(\alpha)-x(\alpha-\beta)|^2}-\frac{1}{|z(\alpha)-z(\alpha-\beta)|^2}\right)d\alpha\\
&+\int(\partial_\alpha^4 x(\alpha)-\partial_\alpha^4 x(\alpha-\beta))^\perp \gamma(\alpha-\beta)\left(\frac{1}{|x(\alpha)-x(\alpha-\beta)|^2}-\frac{1}{|z(\alpha)-z(\alpha-\beta)|^2}\right)d\alpha\\
&+4\int(\partial_\alpha^3 x(\alpha)-\partial_\alpha^3 x(\alpha-\beta))^\perp \partial_\alpha d(\alpha-\beta)\frac{1}{|x(\alpha)-x(\alpha-\beta)|^2}d\alpha\\
&+4\int(\partial_\alpha^3 x(\alpha)-\partial_\alpha^3 x(\alpha-\beta))^\perp \partial_\alpha d(\alpha-\beta)\left(\frac{1}{|x(\alpha)-x(\alpha-\beta)|^2}-\frac{1}{|z(\alpha)-z(\alpha-\beta)|^2}\right)d\alpha\\
&-8\int(\partial_\alpha^3 x(\alpha)-\partial_\alpha^3 x(\alpha-\beta))^\perp d(\alpha-\beta)\left(\frac{1}{|x(\alpha)-x(\alpha-\beta)|^2}\right)^2\\
&\times(\partial_\alpha x(\alpha)-\partial_\alpha x(\alpha-\beta))\cdot(D(\alpha)-D(\alpha-\beta))d\alpha\\
&-8\int(\partial_\alpha^3 x(\alpha)-\partial_\alpha^3 x(\alpha-\beta))^\perp d(\alpha-\beta)\left(\frac{1}{|x(\alpha)-x(\alpha-\beta)|^2}\right)^2\\
&\times(\partial_\alpha x(\alpha)-\partial_\alpha x(\alpha-\beta))\cdot(x(\alpha)-x(\alpha-\beta))d\alpha\\
&-8\int(\partial_\alpha^3 x(\alpha)-\partial_\alpha^3 x(\alpha-\beta))^\perp d(\alpha-\beta)\left(\frac{1}{|x(\alpha)-x(\alpha-\beta)|^2}\right)^2\\
&\times(\partial_\alpha D(\alpha)-\partial_\alpha D(\alpha-\beta))\cdot(D(\alpha)-D(\alpha-\beta))d\alpha\\
&-8\int(\partial_\alpha^3 x(\alpha)-\partial_\alpha^3 x(\alpha-\beta))^\perp d(\alpha-\beta)\left(\frac{1}{|x(\alpha)-x(\alpha-\beta)|^2}\right)^2\\
&\times(x(\alpha)-x(\alpha-\beta))\cdot(\partial_\alpha D(\alpha)-\partial_\alpha D(\alpha-\beta))d\alpha\\
&-8\int(\partial_\alpha^3 x(\alpha)-\partial_\alpha^3 x(\alpha-\beta))^\perp d(\alpha-\beta)\left(\left(\frac{1}{|x(\alpha)-x(\alpha-\beta)|^2}\right)^2-\left(\frac{1}{|z(\alpha)-z(\alpha-\beta)|^2}\right)^2\right)\\
&\times(\partial_\alpha D(\alpha)-\partial_\alpha D(\alpha-\beta))\cdot(D(\alpha)-D(\alpha-\beta))d\alpha\\
&+2831\text{ more terms...}
\end{aligned}
$$

However, there is a significant way to reduce the number of terms in the estimates: writing the equation in complex form instead of vector form. Thus, we can write the evolution for z in the following way:

$$
\partial_t z^*(\alpha,t)=\frac{1}{2\pi}\int_{\mathbb{T}}\frac{1}{z(\alpha,t)-z(\beta,t)}\omega(\beta,t)d\beta+c(\alpha,t)\partial_\alpha z^*(\alpha,t)
$$

In this formulation, the amount of terms of the fourth derivative accounts for only 140 terms. We present the first 10 below.

$$
\begin{aligned}
&2\pi\left(\partial_\alpha^4 BR(x,\gamma)-\partial_\alpha^4 BR(z,\omega)\right)\\
&=-72\int(\partial_\alpha^2 x(\alpha)-\partial_\alpha^2 x(\alpha-\beta))(\partial_\alpha x(\alpha)-\partial_\alpha x(\alpha-\beta))\left(\frac{1}{x(\alpha)-x(\alpha-\beta)}\right)^4\\
&\times(\partial_\alpha D(\alpha)-\partial_\alpha D(\alpha-\beta))d(\alpha-\beta)d\alpha\\
&-72\int(\partial_\alpha^2 x(\alpha)-\partial_\alpha^2 x(\alpha-\beta))(\partial_\alpha x(\alpha)-\partial_\alpha x(\alpha-\beta))\left(\frac{1}{x(\alpha)-x(\alpha-\beta)}\right)^4\\
&\times(\partial_\alpha D(\alpha)-\partial_\alpha D(\alpha-\beta))\gamma(\alpha-\beta)d\alpha\\
&-72\int(\partial_\alpha x(\alpha)-\partial_\alpha x(\alpha-\beta))(\partial_\alpha^2 D(\alpha)-\partial_\alpha^2 D(\alpha-\beta))\left(\frac{1}{x(\alpha)-x(\alpha-\beta)}\right)^4\\
&\times(\partial_\alpha D(\alpha)-\partial_\alpha D(\alpha-\beta))d(\alpha-\beta)d\alpha\\
&-72\int(\partial_\alpha x(\alpha)-\partial_\alpha x(\alpha-\beta))(\partial_\alpha^2 D(\alpha)-\partial_\alpha^2 D(\alpha-\beta))\left(\frac{1}{x(\alpha)-x(\alpha-\beta)}\right)^4\\
&\times(\partial_\alpha D(\alpha)-\partial_\alpha D(\alpha-\beta))\gamma(\alpha-\beta)d\alpha\\
&-36\int(\partial_\alpha^2 D(\alpha)-\partial_\alpha^2 D(\alpha-\beta))\left(\partial_\alpha D(\alpha)-\partial_\alpha D(\alpha-\beta)\right)^2\left(\frac{1}{x(\alpha)-x(\alpha-\beta)}\right)^4 d(\alpha-\beta)d\alpha\\
&-36\int(\partial_\alpha^2 D(\alpha)-\partial_\alpha^2 D(\alpha-\beta))\left(\partial_\alpha D(\alpha)-\partial_\alpha D(\alpha-\beta)\right)^2\left(\frac{1}{x(\alpha)-x(\alpha-\beta)}\right)^4 \gamma(\alpha-\beta)d\alpha\\
&+8\int\left(\frac{1}{x(\alpha)-x(\alpha-\beta)}\right)^3(\partial_\alpha^3 D(\alpha)-\partial_\alpha^3 D(\alpha-\beta))(\partial_\alpha D(\alpha)-\partial_\alpha D(\alpha-\beta))d(\alpha-\beta)d\alpha\\
&+8\int\left(\frac{1}{x(\alpha)-x(\alpha-\beta)}\right)^3(\partial_\alpha^3 D(\alpha)-\partial_\alpha^3 D(\alpha-\beta))(\partial_\alpha D(\alpha)-\partial_\alpha D(\alpha-\beta))\gamma(\alpha-\beta)d\alpha\\
&+24\int\left(\frac{1}{x(\alpha)-x(\alpha-\beta)}\right)^3(\partial_\alpha^2 D(\alpha)-\partial_\alpha^2 D(\alpha-\beta))(\partial_\alpha D(\alpha)-\partial_\alpha D(\alpha-\beta))\partial_\alpha d(\alpha-\beta)d\alpha\\
&+24\int\left(\frac{1}{x(\alpha)-x(\alpha-\beta)}\right)^3(\partial_\alpha^2 D(\alpha)-\partial_\alpha^2 D(\alpha-\beta))(\partial_\alpha D(\alpha)-\partial_\alpha D(\alpha-\beta))\partial_\alpha\gamma(\alpha-\beta)d\alpha\\
&+130\text{ more terms...}
\end{aligned}
$$

The final observation is that if we consider $\mathcal{E}(t)$ as a scalar, we might not get suitable estimates. In order to get better estimates, we will modify the energy into a "vectorized" version $\mathcal{E}_v(t)$, which we will also denote by $\mathcal{E}(t)$ by abuse of notation. This new vectorized energy will be as follows

$$\mathcal{E}(t) = \begin{pmatrix} \|D\|_{L^2} \\ \|D\|_{\dot{H}^1} \\ \|D\|_{\dot{H}^2} \\ \|D\|_{\dot{H}^3} \\ \|d\|_{L^2} \\ \|d\|_{\dot{H}^1} \\ \|d\|_{\dot{H}^2} \\ \vdots \end{pmatrix},$$

where the inhomogeneous spaces $\dot{H}^k$ have their norm defined by $\|f\|_{\dot{H}^k} = \|\partial_\alpha^k f\|_{L^2}$. With this vectorized system, we avoid both the bounding of any given norm by the full energy and any constant factor arising from interpolation between two Sobolev spaces. Thus, our constant $C(t)$ will roughly be of a size comparable to the largest eigenvalue of the linearized system.

5.3.2 Estimates for the linear terms with $Q = 1$

Since we expect $\mathcal{E}(t)$ to be small, the terms that affect more to the evolution of $\mathcal{E}(t)$ are the linear ones. We now report on the non-rigorous experiments over the linear terms to obtain an approximate bound of the behavior of the full system (i.e. an approximation to the largest eigenvalue of the linearized system). We remark that a multiplication of the estimates by a constant, even a small factor 2 for example, has a big impact on the system, rendering the estimates useless and the estimations not tight enough, because the type of estimates we are going to get are exponential on the product of the time elapsed between the splash and the graph and the constant. Therefore, we should be very careful and fine estimates have to be developed.

First of all, we will work with $Q = 1$ and later move on to the case $Q \neq 1$. We will adopt the following convention do denote the different Kernels (integral operators) that appear:

$$\begin{aligned}
\Theta^{a_1,a_2,a_3,a_4}_{b_1,b_2}(\alpha,\beta) &= \frac{1}{(x(\alpha)-x(\beta))^{b_1}}(\partial_\alpha x(\alpha)-\partial_\alpha x(\beta))^{a_1}(\partial_\alpha^2 x(\alpha)-\partial_\alpha^2 x(\beta))^{a_2} \\
&\quad \times (\partial_\alpha^3 x(\alpha)-\partial_\alpha^3 x(\beta))^{a_3}(\partial_\alpha^4 x(\alpha)-\partial_\alpha^4 x(\beta))^{a_4}\partial_\alpha^{b_2}\gamma(\beta) \\
\Theta^{a_1,a_2,a_3,a_4}_{b_1,-1}(\alpha,\beta) &= \frac{1}{(x(\alpha)-x(\beta))^{b_1}}(\partial_\alpha x(\alpha)-\partial_\alpha x(\beta))^{a_1}(\partial_\alpha^2 x(\alpha)-\partial_\alpha^2 x(\beta))^{a_2} \\
&\quad \times (\partial_\alpha^3 x(\alpha)-\partial_\alpha^3 x(\beta))^{a_3}(\partial_\alpha^4 x(\alpha)-\partial_\alpha^4 x(\beta))^{a_4}.
\end{aligned}$$

The operators for which $b_2 \neq -1$ will act on D or its derivatives whereas the operators for which $b_2 = -1$ will act on d or its derivatives. We now describe how to split the Kernels in such a way that they can be computed. For the case where $b_2 \neq -1$ we illustrate this by splitting $\Theta^{0,0,0,0}_{2,0}$, but the technique can be applied to any Kernel.

$$\frac{1}{2\pi}\int \Theta^{0,0,0,0}_{2,0}(D(\alpha)-D(\beta))d\beta = \underbrace{\frac{1}{2\pi}D(\alpha)\int K(\alpha,\beta)\gamma(\beta)d\beta}_{T_1} - \underbrace{\frac{1}{2\pi}\int K(\alpha,\beta)\gamma(\beta)D(\beta)d\beta}_{T_2}$$
$$+\underbrace{\frac{1}{2\pi}c_1(\alpha)\int \frac{D(\alpha)-D(\beta)}{4\sin^2\left(\frac{\alpha-\beta}{2}\right)}\gamma(\beta)d\beta}_{T_3} + \underbrace{\frac{1}{2\pi}c_2(\alpha)\int \frac{D(\alpha)-D(\beta)}{2\tan\left(\frac{\alpha-\beta}{2}\right)}\gamma(\beta)d\beta}_{T_4}, \quad (5.8)$$

where

$$\begin{aligned} K(\alpha,\beta) &= \frac{1}{(x(\alpha)-x(\beta))^2} - \frac{c_1(\alpha)}{4\sin^2\left(\frac{\alpha-\beta}{2}\right)} - \frac{c_2(\alpha)}{2\tan\left(\frac{\alpha-\beta}{2}\right)} \\ c_1(\alpha) &= \frac{1}{x_\alpha^2(\alpha)} \\ c_2(\alpha) &= \frac{x_{\alpha\alpha}(\alpha)}{x_\alpha^3(\alpha)}. \end{aligned}$$

We can think of $c_1(\alpha)$ and $c_2(\alpha)$ as the Taylor coefficients of $\Theta(\alpha,\beta)$ around $\beta=\alpha$. We can bound the terms in (5.8) in the following way:

$$\begin{aligned} T_4(\alpha) &= c_2(\alpha)[H(D\gamma)(\alpha) - DH(\gamma)(\alpha)] \\ T_3(\alpha) &= c_1(\alpha)[\Lambda(D\gamma)(\alpha) - D\Lambda(\gamma)(\alpha)] \end{aligned}$$

We have then the estimates

$$\begin{aligned} \|T_4\|_{L^2} &\leq \|c_2\|_{L^\infty}(\|D\|_{L^2}\|\gamma\|_{L^\infty} + \|D\|_{L^2}\|H\gamma\|_{L^\infty}) \\ \|T_3\|_{L^2} &\leq \|c_1\|_{L^\infty}(\|D\|_{L^2}\|\gamma_\alpha\|_{L^\infty} + \|D_\alpha\|_{L^2}\|\gamma\|_{L^\infty} + \|D\|_{L^2}\|\Lambda(\gamma)\|_{L^\infty}). \end{aligned}$$

We now move on to T_1. We will estimate it in the following way:

$$\int T_1\overline{D(\alpha)}d\alpha = \frac{1}{2\pi}\int |D(\alpha)|^2\int K(\alpha,\beta)\gamma(\beta)d\beta d\alpha \leq \frac{1}{2\pi}\|D\|^2_{L^2}\left\|\int K(\cdot,\beta)\gamma(\beta)d\beta\right\|_{L^\infty}.$$

To estimate the kernel T_2 we will use the Generalized Young's inequality [46]:

$$\|T_2(D)\|^2_{L^2} = \frac{1}{4\pi^2}\int\int\int K(\alpha,\beta)\gamma(\beta)D(\beta)\overline{K(\alpha,\sigma)\gamma(\sigma)D(\sigma)}d\beta d\sigma d\alpha.$$

Defining

$$\tilde{K}(\beta,\sigma) = \int K(\alpha,\beta)\gamma(\beta)\overline{K(\alpha,\sigma)}\gamma(\sigma)d\alpha,$$

we have that

$$\begin{aligned}
\|T_2(D)\|_{L^2}^2 &= \frac{1}{4\pi^2}\int\int \tilde{K}(\beta,\sigma)D(\beta)\overline{D(\sigma)}d\beta d\sigma \\
&= \frac{1}{4\pi^2}\int D(\beta)\left(\int \tilde{K}(\beta,\sigma)\overline{D(\sigma)}d\sigma\right)d\beta \\
&\leq \frac{1}{4\pi^2}\|D\|_{L^2}\left\|\int \tilde{K}(\cdot,\sigma)d\sigma\right\|_{L^2} \\
&\leq \frac{1}{4\pi^2}C\|D\|_{L^2}^2, \quad C = \max\left\{\max_{\beta}\int|\tilde{K}(\beta,\sigma)|d\sigma, \max_{\sigma}\int|\tilde{K}(\beta,\sigma)|d\beta\right\}
\end{aligned}$$

We finally show how to estimate the Kernels with $b_2 = -1$. We will do this by showing how to estimate $\Theta_{1,-1}^{0,0,0,0}$ but the technique can be applied to any Kernel.

$$\begin{aligned}
\frac{1}{2\pi}\int \Theta_{1,-1}^{0,0,0,0}(d(\beta))d\beta = &\underbrace{\frac{1}{2\pi}\int K(\alpha,\beta)d(\beta)d\beta}_{T_1} \\
&+\underbrace{\frac{1}{2\pi}c_1(\alpha)\int\frac{1}{2\tan\left(\frac{\alpha-\beta}{2}\right)}d(\beta)d\beta}_{T_2},
\end{aligned}$$

where

$$\begin{aligned}
K(\alpha,\beta) &= \frac{1}{(x(\alpha)-x(\beta))} - \frac{c_1(\alpha)}{2\tan\left(\frac{\alpha-\beta}{2}\right)} \\
c_1(\alpha) &= \frac{1}{x_\alpha(\alpha)}.
\end{aligned}$$

We can easily estimate these two terms applying to T_1 the same estimates (Young's inequality) as for T_2 in the previous case and by noting that T_2 is $\frac{1}{2}c_1(\alpha)H(d)$.

5.3.3 Estimates for the linear terms with $Q \neq 1$

To perform the real estimates, where $Q \neq 1$ we will use the estimates from the previous sections. We will explain how to pass from the former ones to the latter ones. We will illustrate this by computing the linear terms of the Birkhoff-Rott operator.

First of all, the total number of terms will increase by a factor 2, since we will have

$$\begin{aligned}
Q^2(z)BR(z,\omega) - Q^2(x)BR(x,\gamma)) = &\underbrace{(Q^2(z)-Q^2(x))(BR(z,\omega)-BR(x,\gamma))}_{\text{nonlinear}} \\
&+\underbrace{Q^2(x)(BR(z,\omega)-BR(x,\gamma))}_{\text{calculated before}} \\
&+\underbrace{(Q^2(z)-Q^2(x))BR(x,\gamma)}_{\text{new terms}}
\end{aligned}$$

In order to calculate the old terms with $Q \neq 1$, the only thing we have to do is to incorporate a factor of $\partial_\alpha^k Q^2(x)(\alpha)$ on the estimates. The new terms can easily be calculated using that, up to linear order

$$(Q^2(z) - Q^2(x)) = \frac{1}{8}\left\langle \frac{1+x^4}{x}, \overline{3x^2 - \frac{1}{x^2}} \right\rangle D + O(D^2).$$

The following tables summarize the (non-rigorous) estimations of the kernels obtained for the numerical simulations at time $t = 0$. We will mean by "acting on" $D(\alpha) - D(\beta)$ or its derivatives whenever D is referred, and $d(\beta)$ or its derivatives whenever d is referred.

5.3.3.1 0 derivatives in Q: Linear terms

Num	Kernel	Acts	T_1	T_2	$T_{3,1}$	$T_{3,2}$	$T_{3,3}$	$T_{4,1}$	$T_{4,2}$
1	$\Theta^{0,0,0,0}_{1,-1}$	d	$1.14 \cdot 10^0$	$2.01 \cdot 10^0$					
2	$\Theta^{0,0,0,0}_{2,0}$	D	$2.54 \cdot 10^3$	$8.47 \cdot 10^2$	$6.08 \cdot 10^3$	$2.69 \cdot 10^1$	$4.31 \cdot 10^3$	$3.59 \cdot 10^3$	$9.03 \cdot 10^3$
1	$\Theta^{0,0,0,0}_{2,0}$	D_α	$2.54 \cdot 10^3$	$8.47 \cdot 10^2$	$6.08 \cdot 10^3$	$2.69 \cdot 10^1$	$4.31 \cdot 10^3$	$3.59 \cdot 10^3$	$9.03 \cdot 10^3$
2	$\Theta^{0,0,0,0}_{1,-1}$	d_α	$1.14 \cdot 10^0$	$2.01 \cdot 10^0$					
3	$\Theta^{0,0,0,0}_{2,1}$	D	$6.16 \cdot 10^4$	$9.32 \cdot 10^4$	$2.15 \cdot 10^6$	$6.08 \cdot 10^3$	$2.46 \cdot 10^6$	$6.37 \cdot 10^5$	$1.04 \cdot 10^6$
4	$\Theta^{1,0,0,0}_{3,0}$	D	$5.48 \cdot 10^5$	$2.05 \cdot 10^5$	$1.05 \cdot 10^6$	$3.60 \cdot 10^3$	$6.37 \cdot 10^5$	$1.22 \cdot 10^6$	$3.12 \cdot 10^6$
5	$\Theta^{1,0,0,0}_{2,-1}$	d	$2.42 \cdot 10^1$	$3.51 \cdot 10^1$					
1	$\Theta^{1,0,0,0}_{3,0}$	D_α	$5.48 \cdot 10^5$	$2.05 \cdot 10^5$	$1.05 \cdot 10^6$	$3.60 \cdot 10^3$	$6.37 \cdot 10^5$	$1.22 \cdot 10^6$	$3.12 \cdot 10^6$
2	$\Theta^{0,0,0,0}_{2,0}$	$D_{\alpha\alpha}$	$2.54 \cdot 10^3$	$8.47 \cdot 10^2$	$6.08 \cdot 10^3$	$2.69 \cdot 10^1$	$4.31 \cdot 10^3$	$3.59 \cdot 10^3$	$9.03 \cdot 10^3$
3	$\Theta^{0,0,0,0}_{2,1}$	D_α	$6.16 \cdot 10^4$	$9.32 \cdot 10^4$	$2.15 \cdot 10^6$	$6.08 \cdot 10^3$	$2.46 \cdot 10^6$	$6.37 \cdot 10^5$	$1.04 \cdot 10^6$
4	$\Theta^{0,1,0,0}_{2,-1}$	d	$5.41 \cdot 10^5$	$6.59 \cdot 10^3$					
5	$\Theta^{1,0,0,0}_{2,-1}$	d_α	$2.42 \cdot 10^1$	$3.51 \cdot 10^1$					
6	$\Theta^{2,0,0,0}_{3,-1}$	d	$4.36 \cdot 10^3$	$6.10 \cdot 10^3$					
7	$\Theta^{0,0,0,0}_{1,-1}$	$d_{\alpha\alpha}$	$1.14 \cdot 10^0$	$2.01 \cdot 10^0$					
8	$\Theta^{0,1,0,0}_{3,0}$	D	$8.71 \cdot 10^7$	$3.60 \cdot 10^7$	$1.83 \cdot 10^8$	$7.91 \cdot 10^5$	$1.36 \cdot 10^8$	$8.08 \cdot 10^7$	$1.90 \cdot 10^8$
9	$\Theta^{1,0,0,0}_{3,1}$	D	$2.23 \cdot 10^7$	$2.40 \cdot 10^7$	$3.35 \cdot 10^8$	$1.05 \cdot 10^6$	$4.25 \cdot 10^8$	$3.62 \cdot 10^8$	$2.15 \cdot 10^8$
10	$\Theta^{2,0,0,0}_{4,0}$	D	$5.20 \cdot 10^7$	$3.72 \cdot 10^7$	$1.81 \cdot 10^8$	$5.70 \cdot 10^5$	$9.91 \cdot 10^7$	$1.15 \cdot 10^8$	$2.72 \cdot 10^8$
11	$\Theta^{0,0,0,0}_{2,2}$	D	$7.75 \cdot 10^6$	$3.11 \cdot 10^7$	$1.94 \cdot 10^9$	$2.15 \cdot 10^6$	$1.51 \cdot 10^9$	$3.35 \cdot 10^8$	$4.25 \cdot 10^8$
1	$\Theta^{0,1,0,0}_{3,0}$	D_α	$8.71 \cdot 10^7$	$3.60 \cdot 10^7$	$1.83 \cdot 10^8$	$7.91 \cdot 10^5$	$1.36 \cdot 10^8$	$8.08 \cdot 10^7$	$1.90 \cdot 10^8$
2	$\Theta^{1,1,0,0}_{3,-1}$	d	$9.25 \cdot 10^5$	$1.06 \cdot 10^6$					
3	$\Theta^{1,0,0,0}_{3,0}$	$D_{\alpha\alpha}$	$5.48 \cdot 10^5$	$2.05 \cdot 10^5$	$1.05 \cdot 10^6$	$3.60 \cdot 10^3$	$6.37 \cdot 10^5$	$1.22 \cdot 10^6$	$3.12 \cdot 10^6$
4	$\Theta^{1,0,0,0}_{3,1}$	D_α	$2.23 \cdot 10^7$	$2.40 \cdot 10^7$	$3.35 \cdot 10^8$	$1.05 \cdot 10^6$	$4.25 \cdot 10^8$	$3.62 \cdot 10^8$	$2.15 \cdot 10^8$
5	$\Theta^{2,0,0,0}_{4,0}$	D_α	$5.20 \cdot 10^7$	$3.72 \cdot 10^7$	$1.81 \cdot 10^8$	$5.70 \cdot 10^5$	$9.91 \cdot 10^7$	$1.15 \cdot 10^8$	$2.72 \cdot 10^8$
6	$\Theta^{0,0,0,0}_{2,0}$	$\partial^3_\alpha D$	$2.54 \cdot 10^3$	$8.47 \cdot 10^2$	$6.08 \cdot 10^3$	$2.69 \cdot 10^1$	$4.31 \cdot 10^3$	$3.59 \cdot 10^3$	$9.03 \cdot 10^3$
7	$\Theta^{0,0,0,0}_{2,1}$	$D_{\alpha\alpha}$	$6.16 \cdot 10^4$	$9.32 \cdot 10^4$	$2.15 \cdot 10^6$	$6.08 \cdot 10^3$	$2.46 \cdot 10^6$	$6.37 \cdot 10^5$	$1.04 \cdot 10^6$
8	$\Theta^{0,0,0,0}_{2,2}$	D_α	$7.75 \cdot 10^6$	$3.11 \cdot 10^7$	$1.94 \cdot 10^9$	$2.15 \cdot 10^6$	$1.51 \cdot 10^9$	$3.35 \cdot 10^8$	$4.25 \cdot 10^8$
9	$\Theta^{0,0,1,0}_{2,-1}$	d	$3.78 \cdot 10^6$	$4.65 \cdot 10^6$					
10	$\Theta^{0,1,0,0}_{2,-1}$	d_α	$5.41 \cdot 10^5$	$6.59 \cdot 10^3$					

11	$\Theta_{4,0}^{1,1,0,0}$	D	$1.52 \cdot 10^{10}$	$6.68 \cdot 10^{9}$	$3.13 \cdot 10^{10}$	$1.15 \cdot 10^{8}$	$2.04 \cdot 10^{10}$	$1.73 \cdot 10^{10}$	$3.26 \cdot 10^{10}$
12	$\Theta_{2,-1}^{1,0,0,0}$	$d_{\alpha\alpha}$	$2.42 \cdot 10^{1}$	$3.51 \cdot 10^{1}$					
13	$\Theta_{3,-1}^{2,0,0,0}$	d_{α}	$4.36 \cdot 10^{3}$	$6.10 \cdot 10^{3}$					
14	$\Theta_{4,-1}^{3,0,0,0}$	d	$7.73 \cdot 10^{5}$	$1.05 \cdot 10^{6}$					
15	$\Theta_{1,-1}^{0,0,0,0}$	$\partial_{\alpha}^{3} d$	$1.14 \cdot 10^{0}$	$2.01 \cdot 10^{0}$					
16	$\Theta_{3,0}^{0,0,1,0}$	D	$1.48 \cdot 10^{11}$	$6.97 \cdot 10^{10}$	$1.38 \cdot 10^{11}$	$4.08 \cdot 10^{8}$	$6.73 \cdot 10^{10}$	$1.29 \cdot 10^{11}$	$3.00 \cdot 10^{11}$
17	$\Theta_{3,1}^{0,1,0,0}$	D	$4.43 \cdot 10^{9}$	$7.99 \cdot 10^{9}$	$6.87 \cdot 10^{10}$	$1.83 \cdot 10^{8}$	$7.35 \cdot 10^{10}$	$1.81 \cdot 10^{8}$	$1.08 \cdot 10^{8}$
18	$\Theta_{3,2}^{1,0,0,0}$	D	$2.87 \cdot 10^{9}$	$8.14 \cdot 10^{9}$	$3.35 \cdot 10^{11}$	$3.35 \cdot 10^{8}$	$2.42 \cdot 10^{11}$	$1.14 \cdot 10^{11}$	$1.47 \cdot 10^{11}$
19	$\Theta_{4,1}^{2,0,0,0}$	D	$5.63 \cdot 10^{9}$	$7.13 \cdot 10^{9}$	$5.38 \cdot 10^{10}$	$1.81 \cdot 10^{8}$	$7.33 \cdot 10^{10}$	$3.13 \cdot 10^{10}$	$2.04 \cdot 10^{10}$
20	$\Theta_{5,0}^{3,0,0,0}$	D	$2.27 \cdot 10^{10}$	$8.52 \cdot 10^{9}$	$3.12 \cdot 10^{10}$	$9.40 \cdot 10^{7}$	$1.59 \cdot 10^{10}$	$2.22 \cdot 10^{10}$	$5.11 \cdot 10^{10}$
21	$\Theta_{2,3}^{0,0,0,0}$	D	$1.51 \cdot 10^{9}$	$2.33 \cdot 10^{10}$	$1.69 \cdot 10^{12}$	$1.95 \cdot 10^{9}$	$1.75 \cdot 10^{12}$	$3.35 \cdot 10^{11}$	$2.42 \cdot 10^{11}$
1	$\Theta_{4,0}^{1,1,0,0}$	D_{α}	$1.52 \cdot 10^{10}$	$6.68 \cdot 10^{9}$	$3.13 \cdot 10^{10}$	$1.15 \cdot 10^{8}$	$2.04 \cdot 10^{10}$	$1.73 \cdot 10^{10}$	$3.26 \cdot 10^{10}$
2	$\Theta_{3,0}^{0,0,1,0}$	D_{α}	$1.48 \cdot 10^{11}$	$6.97 \cdot 10^{10}$	$1.38 \cdot 10^{11}$	$4.08 \cdot 10^{8}$	$6.73 \cdot 10^{10}$	$1.29 \cdot 10^{11}$	$3.00 \cdot 10^{11}$
3	$\Theta_{3,-1}^{1,0,1,0}$	d	$1.79 \cdot 10^{10}$	$4.97 \cdot 10^{6}$					
4	$\Theta_{3,0}^{0,1,0,0}$	$D_{\alpha\alpha}$	$8.71 \cdot 10^{7}$	$3.60 \cdot 10^{7}$	$1.83 \cdot 10^{8}$	$7.91 \cdot 10^{5}$	$1.36 \cdot 10^{8}$	$8.08 \cdot 10^{7}$	$1.90 \cdot 10^{8}$
5	$\Theta_{3,1}^{0,1,0,0}$	D_{α}	$4.43 \cdot 10^{9}$	$7.99 \cdot 10^{9}$	$6.87 \cdot 10^{10}$	$1.83 \cdot 10^{8}$	$7.35 \cdot 10^{10}$	$1.81 \cdot 10^{8}$	$1.08 \cdot 10^{8}$
6	$\Theta_{3,-1}^{1,1,0,0}$	d_{α}	$9.25 \cdot 10^{5}$	$1.06 \cdot 10^{6}$					
7	$\Theta_{4,-1}^{2,1,0,0}$	d	$1.55 \cdot 10^{8}$	$1.82 \cdot 10^{8}$					
8	$\Theta_{3,0}^{1,0,0,0}$	$\partial_{\alpha}^{3} D$	$5.48 \cdot 10^{5}$	$2.05 \cdot 10^{5}$	$1.05 \cdot 10^{6}$	$3.60 \cdot 10^{3}$	$6.37 \cdot 10^{5}$	$1.22 \cdot 10^{6}$	$3.12 \cdot 10^{6}$
9	$\Theta_{3,1}^{1,0,0,0}$	$D_{\alpha\alpha}$	$2.23 \cdot 10^{7}$	$2.40 \cdot 10^{7}$	$3.35 \cdot 10^{8}$	$1.05 \cdot 10^{6}$	$4.25 \cdot 10^{8}$	$3.62 \cdot 10^{8}$	$2.15 \cdot 10^{8}$
10	$\Theta_{3,2}^{1,0,0,0}$	D_{α}	$2.87 \cdot 10^{9}$	$8.14 \cdot 10^{9}$	$3.35 \cdot 10^{11}$	$3.35 \cdot 10^{8}$	$2.42 \cdot 10^{11}$	$1.14 \cdot 10^{11}$	$1.47 \cdot 10^{11}$
11	$\Theta_{4,0}^{2,0,0,0}$	$D_{\alpha\alpha}$	$5.20 \cdot 10^{7}$	$3.72 \cdot 10^{7}$	$1.81 \cdot 10^{8}$	$5.70 \cdot 10^{5}$	$9.91 \cdot 10^{7}$	$1.15 \cdot 10^{8}$	$2.72 \cdot 10^{8}$
12	$\Theta_{4,1}^{2,0,0,0}$	D_{α}	$5.63 \cdot 10^{9}$	$7.13 \cdot 10^{9}$	$5.38 \cdot 10^{10}$	$1.81 \cdot 10^{8}$	$7.33 \cdot 10^{10}$	$3.13 \cdot 10^{10}$	$2.04 \cdot 10^{10}$
13	$\Theta_{5,0}^{3,0,0,0}$	D_{α}	$2.27 \cdot 10^{10}$	$8.52 \cdot 10^{9}$	$3.12 \cdot 10^{10}$	$9.40 \cdot 10^{7}$	$1.59 \cdot 10^{10}$	$2.22 \cdot 10^{10}$	$5.11 \cdot 10^{10}$
14	Estimated using an extra cancelation								
15	$\Theta_{2,1}^{0,0,0,0}$	$\partial_{\alpha}^{3} D$	$6.16 \cdot 10^{4}$	$9.32 \cdot 10^{4}$	$2.15 \cdot 10^{6}$	$6.08 \cdot 10^{3}$	$2.46 \cdot 10^{6}$	$6.37 \cdot 10^{5}$	$1.04 \cdot 10^{6}$
16	$\Theta_{2,2}^{0,0,0,0}$	$D_{\alpha\alpha}$	$7.75 \cdot 10^{6}$	$3.11 \cdot 10^{7}$	$1.94 \cdot 10^{9}$	$2.15 \cdot 10^{6}$	$1.51 \cdot 10^{9}$	$3.35 \cdot 10^{8}$	$4.25 \cdot 10^{8}$
17	$\Theta_{2,3}^{0,0,0,0}$	D_{α}	$1.51 \cdot 10^{9}$	$2.33 \cdot 10^{10}$	$1.69 \cdot 10^{12}$	$1.95 \cdot 10^{9}$	$1.75 \cdot 10^{12}$	$3.35 \cdot 10^{11}$	$2.42 \cdot 10^{11}$
18	$\Theta_{2,-1}^{0,0,0,1}$	d	$3.20 \cdot 10^{9}$	$2.86 \cdot 10^{9}$					
19	$\Theta_{2,-1}^{0,0,1,0}$	d_{α}	$3.78 \cdot 10^{6}$	$4.65 \cdot 10^{6}$					

20	$\Theta_{4,0}^{1,0,1,0}$	D	$1.93 \cdot 10^{13}$	$6.63 \cdot 10^{12}$	$2.37 \cdot 10^{13}$	$6.88 \cdot 10^{10}$	$1.11 \cdot 10^{13}$	$1.77 \cdot 10^{13}$	$4.01 \cdot 10^{13}$
21	$\Theta_{2,-1}^{0,1,0,0}$	$d_{\alpha\alpha}$	$5.41 \cdot 10^{5}$	$6.59 \cdot 10^{3}$					
22	$\Theta_{4,1}^{1,1,0,0}$	D	$9.79 \cdot 10^{11}$	$9.59 \cdot 10^{11}$	$1.07 \cdot 10^{13}$	$3.13 \cdot 10^{10}$	$1.27 \cdot 10^{13}$	$3.79 \cdot 10^{12}$	$2.69 \cdot 10^{12}$
23	$\Theta_{5,0}^{2,1,0,0}$	D	$2.00 \cdot 10^{13}$	$1.03 \cdot 10^{13}$	$5.38 \cdot 10^{12}$	$1.81 \cdot 10^{10}$	$3.16 \cdot 10^{12}$	$3.39 \cdot 10^{12}$	$6.71 \cdot 10^{12}$
24	$\Theta_{2,-1}^{1,0,0,0}$	$\partial_\alpha^3 d$	$2.42 \cdot 10^{1}$	$3.51 \cdot 10^{1}$					
25	$\Theta_{3,-1}^{0,2,0,0}$	d	$2.61 \cdot 10^{8}$	$2.10 \cdot 10^{8}$					
26	$\Theta_{3,-1}^{2,0,0,0}$	$d_{\alpha\alpha}$	$4.36 \cdot 10^{3}$	$6.10 \cdot 10^{3}$					
27	$\Theta_{4,-1}^{3,0,0,0}$	d_α	$7.73 \cdot 10^{5}$	$1.05 \cdot 10^{6}$					
28	$\Theta_{5,-1}^{4,0,0,0}$	d	$1.36 \cdot 10^{8}$	$1.81 \cdot 10^{8}$					
29	$\Theta_{1,-1}^{0,0,0,0}$	$\partial_\alpha^4 d$	$1.14 \cdot 10^{0}$	$2.01 \cdot 10^{0}$					
30	$\Theta_{3,0}^{0,0,0,1}$	D	$1.67 \cdot 10^{14}$	$7.42 \cdot 10^{13}$	$8.15 \cdot 10^{13}$	$3.04 \cdot 10^{11}$	$5.38 \cdot 10^{13}$	$1.19 \cdot 10^{14}$	$3.68 \cdot 10^{14}$
31	$\Theta_{3,1}^{0,0,1,0}$	D	$8.74 \cdot 10^{12}$	$1.06 \cdot 10^{13}$	$3.69 \cdot 10^{13}$	$1.38 \cdot 10^{11}$	$5.59 \cdot 10^{13}$	$3.19 \cdot 10^{13}$	$2.28 \cdot 10^{13}$
32	$\Theta_{3,2}^{0,1,0,0}$	D	$4.12 \cdot 10^{12}$	$3.02 \cdot 10^{12}$	$5.80 \cdot 10^{13}$	$6.88 \cdot 10^{10}$	$4.78 \cdot 10^{13}$	$5.72 \cdot 10^{10}$	$7.34 \cdot 10^{10}$
33	$\Theta_{3,3}^{1,0,0,0}$	D	$6.48 \cdot 10^{11}$	$6.17 \cdot 10^{12}$	$2.78 \cdot 10^{14}$	$3.35 \cdot 10^{11}$	$3.02 \cdot 10^{14}$	$1.16 \cdot 10^{14}$	$8.25 \cdot 10^{13}$
34	$\Theta_{4,0}^{0,2,0,0}$	D	$7.93 \cdot 10^{12}$	$2.78 \cdot 10^{12}$	$5.55 \cdot 10^{12}$	$2.50 \cdot 10^{10}$	$4.36 \cdot 10^{12}$	$6.42 \cdot 10^{12}$	$1.93 \cdot 10^{13}$
35	$\Theta_{4,2}^{2,0,0,0}$	D	$2.52 \cdot 10^{12}$	$2.58 \cdot 10^{12}$	$5.78 \cdot 10^{13}$	$5.38 \cdot 10^{10}$	$3.90 \cdot 10^{13}$	$1.07 \cdot 10^{13}$	$1.27 \cdot 10^{13}$
36	$\Theta_{5,1}^{3,0,0,0}$	D	$1.33 \cdot 10^{12}$	$1.07 \cdot 10^{12}$	$8.64 \cdot 10^{12}$	$3.12 \cdot 10^{10}$	$1.26 \cdot 10^{13}$	$5.58 \cdot 10^{12}$	$3.92 \cdot 10^{12}$
37	$\Theta_{6,0}^{4,0,0,0}$	D	$4.48 \cdot 10^{12}$	$1.63 \cdot 10^{12}$	$5.37 \cdot 10^{12}$	$1.58 \cdot 10^{10}$	$2.58 \cdot 10^{12}$	$4.20 \cdot 10^{12}$	$9.76 \cdot 10^{12}$
38	Estimated using an extra cancelation								

5.3.3.2 1 derivative in Q: Linear terms

Num	Kernel	Acts	T_1	T_2	$T_{3,1}$	$T_{3,2}$	$T_{3,3}$	$T_{4,1}$	$T_{4,2}$
1	$\Theta_{1,-1}^{0,0,0,0}$	d	$3.47 \cdot 10^{0}$	$5.08 \cdot 10^{0}$					
2	$\Theta_{2,0}^{0,0,0,0}$	D	$9.28 \cdot 10^{3}$	$2.91 \cdot 10^{3}$	$2.25 \cdot 10^{4}$	$1.03 \cdot 10^{2}$	$1.13 \cdot 10^{4}$	$9.53 \cdot 10^{3}$	$3.46 \cdot 10^{4}$
1	$\Theta_{2,0}^{0,0,0,0}$	D_α	$9.28 \cdot 10^{3}$	$2.91 \cdot 10^{3}$	$2.25 \cdot 10^{4}$	$1.03 \cdot 10^{2}$	$1.13 \cdot 10^{4}$	$9.53 \cdot 10^{3}$	$3.46 \cdot 10^{4}$
2	$\Theta_{1,-1}^{0,0,0,0}$	d_α	$3.47 \cdot 10^{0}$	$5.08 \cdot 10^{0}$					
3	$\Theta_{2,1}^{0,0,0,0}$	D	$2.71 \cdot 10^{5}$	$3.35 \cdot 10^{5}$	$7.01 \cdot 10^{6}$	$2.25 \cdot 10^{4}$	$8.72 \cdot 10^{6}$	$3.80 \cdot 10^{6}$	$1.54 \cdot 10^{6}$
4	$\Theta_{3,0}^{1,0,0,0}$	D	$2.12 \cdot 10^{6}$	$7.25 \cdot 10^{5}$	$3.80 \cdot 10^{6}$	$9.53 \cdot 10^{3}$	$1.54 \cdot 10^{6}$	$3.28 \cdot 10^{6}$	$1.18 \cdot 10^{7}$
5	$\Theta_{2,-1}^{1,0,0,0}$	d	$1.07 \cdot 10^{2}$	$1.36 \cdot 10^{2}$					

1	$\Theta_{3,0}^{1,0,0,0}$	D_α	$2.12 \cdot 10^6$	$7.25 \cdot 10^5$	$3.80 \cdot 10^6$	$9.53 \cdot 10^3$	$1.54 \cdot 10^6$	$3.28 \cdot 10^6$	$1.18 \cdot 10^7$
2	$\Theta_{2,0}^{0,0,0,0}$	$D_{\alpha\alpha}$	$9.28 \cdot 10^3$	$2.91 \cdot 10^3$	$2.25 \cdot 10^4$	$1.03 \cdot 10^2$	$1.13 \cdot 10^4$	$9.53 \cdot 10^3$	$3.46 \cdot 10^4$
3	$\Theta_{2,1}^{0,0,0,0}$	D_α	$2.71 \cdot 10^5$	$3.35 \cdot 10^5$	$7.01 \cdot 10^6$	$2.25 \cdot 10^4$	$8.72 \cdot 10^6$	$3.80 \cdot 10^6$	$1.54 \cdot 10^6$
4	$\Theta_{2,-1}^{0,1,0,0}$	d	$2.16 \cdot 10^6$	$2.93 \cdot 10^4$					
5	$\Theta_{2,-1}^{1,0,0,0}$	d_α	$1.07 \cdot 10^2$	$1.36 \cdot 10^2$					
6	$\Theta_{3,-1}^{2,0,0,0}$	d	$1.76 \cdot 10^4$	$2.22 \cdot 10^4$					
7	$\Theta_{1,-1}^{0,0,0,0}$	$d_{\alpha\alpha}$	$3.47 \cdot 10^0$	$5.08 \cdot 10^0$					
8	$\Theta_{3,0}^{0,1,0,0}$	D	$3.38 \cdot 10^8$	$1.40 \cdot 10^8$	$6.99 \cdot 10^8$	$1.75 \cdot 10^6$	$3.26 \cdot 10^8$	$1.96 \cdot 10^8$	$7.14 \cdot 10^8$
9	$\Theta_{3,1}^{1,0,0,0}$	D	$7.98 \cdot 10^7$	$8.26 \cdot 10^7$	$1.04 \cdot 10^9$	$3.80 \cdot 10^6$	$1.49 \cdot 10^9$	$1.31 \cdot 10^9$	$5.29 \cdot 10^8$
10	$\Theta_{4,0}^{2,0,0,0}$	D	$2.05 \cdot 10^8$	$1.32 \cdot 10^8$	$6.44 \cdot 10^8$	$1.63 \cdot 10^6$	$2.55 \cdot 10^8$	$2.91 \cdot 10^8$	$1.10 \cdot 10^9$
11	$\Theta_{2,2}^{0,0,0,0}$	D	$2.79 \cdot 10^7$	$1.14 \cdot 10^8$	$6.69 \cdot 10^9$	$7.01 \cdot 10^6$	$5.33 \cdot 10^9$	$1.04 \cdot 10^9$	$1.49 \cdot 10^9$
1	$\Theta_{3,0}^{0,1,0,0}$	D_α	$3.38 \cdot 10^8$	$1.40 \cdot 10^8$	$6.99 \cdot 10^8$	$1.75 \cdot 10^6$	$3.26 \cdot 10^8$	$1.96 \cdot 10^8$	$7.14 \cdot 10^8$
2	$\Theta_{3,-1}^{1,1,0,0}$	d							
3	$\Theta_{3,0}^{1,0,0,0}$	$D_{\alpha\alpha}$	$2.12 \cdot 10^6$	$7.25 \cdot 10^5$	$3.80 \cdot 10^6$	$9.53 \cdot 10^3$	$1.54 \cdot 10^6$	$3.28 \cdot 10^6$	$1.18 \cdot 10^7$
4	$\Theta_{3,1}^{1,0,0,0}$	D_α	$7.98 \cdot 10^7$	$8.26 \cdot 10^7$	$1.04 \cdot 10^9$	$3.80 \cdot 10^6$	$1.49 \cdot 10^9$	$1.31 \cdot 10^9$	$5.29 \cdot 10^8$
5	$\Theta_{4,0}^{2,0,0,0}$	D_α	$2.05 \cdot 10^8$	$1.32 \cdot 10^8$	$6.44 \cdot 10^8$	$1.63 \cdot 10^6$	$2.55 \cdot 10^8$	$2.91 \cdot 10^8$	$1.10 \cdot 10^9$
6	$\Theta_{2,0}^{0,0,0,0}$	$\partial_\alpha^3 D$	$9.28 \cdot 10^3$	$2.91 \cdot 10^3$	$2.25 \cdot 10^4$	$1.03 \cdot 10^2$	$1.13 \cdot 10^4$	$9.53 \cdot 10^3$	$3.46 \cdot 10^4$
7	$\Theta_{2,1}^{0,0,0,0}$	$D_{\alpha\alpha}$	$2.71 \cdot 10^5$	$3.35 \cdot 10^5$	$7.01 \cdot 10^6$	$2.25 \cdot 10^4$	$8.72 \cdot 10^6$	$3.80 \cdot 10^6$	$1.54 \cdot 10^6$
8	$\Theta_{2,2}^{0,0,0,0}$	D_α	$2.79 \cdot 10^7$	$1.14 \cdot 10^8$	$6.69 \cdot 10^9$	$7.01 \cdot 10^6$	$5.33 \cdot 10^9$	$1.04 \cdot 10^9$	$1.49 \cdot 10^9$
9	$\Theta_{2,-1}^{0,0,1,0}$	d	$1.38 \cdot 10^7$	$1.64 \cdot 10^7$					
10	$\Theta_{2,-1}^{0,1,0,0}$	d_α	$2.16 \cdot 10^6$	$2.93 \cdot 10^4$					
11	$\Theta_{4,0}^{1,1,0,0}$	D	$7.09 \cdot 10^{10}$	$2.60 \cdot 10^{10}$	$1.16 \cdot 10^{11}$	$2.91 \cdot 10^8$	$4.87 \cdot 10^{10}$	$3.36 \cdot 10^{10}$	$1.54 \cdot 10^{11}$
12	$\Theta_{2,-1}^{1,0,0,0}$	$d_{\alpha\alpha}$	$1.07 \cdot 10^2$	$1.36 \cdot 10^2$					
13	$\Theta_{3,-1}^{2,0,0,0}$	d_α	$1.76 \cdot 10^4$	$2.22 \cdot 10^4$					
14	$\Theta_{4,-1}^{3,0,0,0}$	d	$3.02 \cdot 10^6$	$3.75 \cdot 10^6$					
15	$\Theta_{1,-1}^{0,0,0,0}$	$\partial_\alpha^3 d$	$3.47 \cdot 10^0$	$5.08 \cdot 10^0$					
16	$\Theta_{3,0}^{0,0,1,0}$	D	$5.92 \cdot 10^{11}$	$2.37 \cdot 10^{11}$	$4.84 \cdot 10^{11}$	$1.23 \cdot 10^9$	$1.81 \cdot 10^{11}$	$3.11 \cdot 10^{11}$	$1.24 \cdot 10^{12}$
17	$\Theta_{3,1}^{0,1,0,0}$	D	$1.67 \cdot 10^{10}$	$2.86 \cdot 10^{10}$	$2.22 \cdot 10^{11}$	$6.99 \cdot 10^8$	$2.62 \cdot 10^{11}$	$6.54 \cdot 10^8$	$2.64 \cdot 10^8$
18	$\Theta_{3,2}^{1,0,0,0}$	D	$1.16 \cdot 10^{10}$	$2.90 \cdot 10^{10}$	$1.15 \cdot 10^{12}$	$1.04 \cdot 10^9$	$8.33 \cdot 10^{11}$	$3.48 \cdot 10^{11}$	$5.15 \cdot 10^{11}$
19	$\Theta_{4,1}^{2,0,0,0}$	D	$1.95 \cdot 10^{10}$	$2.75 \cdot 10^{10}$	$1.56 \cdot 10^{11}$	$6.44 \cdot 10^8$	$2.56 \cdot 10^{11}$	$1.16 \cdot 10^{11}$	$4.87 \cdot 10^{10}$

20	$\Theta_{5,0}^{3,0,0,0}$	D	$9.19 \cdot 10^{10}$	$3.05 \cdot 10^{10}$	$1.10 \cdot 10^{11}$	$2.79 \cdot 10^{8}$	$4.20 \cdot 10^{10}$	$5.43 \cdot 10^{10}$	$2.11 \cdot 10^{11}$
21	$\Theta_{2,3}^{0,0,0,0}$	D	$7.01 \cdot 10^{9}$	$8.38 \cdot 10^{10}$	$6.19 \cdot 10^{12}$	$6.69 \cdot 10^{9}$	$6.02 \cdot 10^{12}$	$1.15 \cdot 10^{12}$	$8.33 \cdot 10^{11}$

5.3.3.3 2 derivatives in Q: Linear terms

Num	Kernel	Acts	T_1	T_2	$T_{3,1}$	$T_{3,2}$	$T_{3,3}$	$T_{4,1}$	$T_{4,2}$
1	$\Theta_{1,-1}^{0,0,0,0}$	d	$1.20 \cdot 10^{2}$	$1.51 \cdot 10^{2}$					
2	$\Theta_{2,0}^{0,0,0,0}$	D	$1.86 \cdot 10^{6}$	$4.73 \cdot 10^{5}$	$4.22 \cdot 10^{6}$	$1.71 \cdot 10^{4}$	$3.02 \cdot 10^{6}$	$2.62 \cdot 10^{6}$	$6.38 \cdot 10^{6}$
1	$\Theta_{2,0}^{0,0,0,0}$	D_α	$1.86 \cdot 10^{6}$	$4.73 \cdot 10^{5}$	$4.22 \cdot 10^{6}$	$1.71 \cdot 10^{4}$	$3.02 \cdot 10^{6}$	$2.62 \cdot 10^{6}$	$6.38 \cdot 10^{6}$
2	$\Theta_{1,-1}^{0,0,0,0}$	d_α	$1.20 \cdot 10^{2}$	$1.51 \cdot 10^{2}$					
3	$\Theta_{2,1}^{0,0,0,0}$	D	$4.19 \cdot 10^{7}$	$4.45 \cdot 10^{7}$	$1.56 \cdot 10^{9}$	$4.22 \cdot 10^{6}$	$1.70 \cdot 10^{9}$	$7.28 \cdot 10^{8}$	$4.62 \cdot 10^{8}$
4	$\Theta_{3,0}^{1,0,0,0}$	D	$3.90 \cdot 10^{8}$	$1.18 \cdot 10^{8}$	$7.28 \cdot 10^{8}$	$2.62 \cdot 10^{6}$	$4.62 \cdot 10^{8}$	$8.92 \cdot 10^{8}$	$2.20 \cdot 10^{9}$
5	$\Theta_{2,-1}^{1,0,0,0}$	d	$1.43 \cdot 10^{4}$	$2.55 \cdot 10^{4}$					
1	$\Theta_{3,0}^{1,0,0,0}$	D_α	$3.90 \cdot 10^{8}$	$1.18 \cdot 10^{8}$	$7.28 \cdot 10^{8}$	$2.62 \cdot 10^{6}$	$4.62 \cdot 10^{8}$	$8.92 \cdot 10^{8}$	$2.20 \cdot 10^{9}$
2	$\Theta_{2,0}^{0,0,0,0}$	$D_{\alpha\alpha}$	$1.86 \cdot 10^{6}$	$4.73 \cdot 10^{5}$	$4.22 \cdot 10^{6}$	$1.71 \cdot 10^{4}$	$3.02 \cdot 10^{6}$	$2.62 \cdot 10^{6}$	$6.38 \cdot 10^{6}$
3	$\Theta_{2,1}^{0,0,0,0}$	D_α	$4.19 \cdot 10^{7}$	$4.45 \cdot 10^{7}$	$1.56 \cdot 10^{9}$	$4.22 \cdot 10^{6}$	$1.70 \cdot 10^{9}$	$7.28 \cdot 10^{8}$	$4.62 \cdot 10^{8}$
4	$\Theta_{2,-1}^{0,1,0,0}$	d	$3.19 \cdot 10^{8}$	$4.77 \cdot 10^{6}$					
5	$\Theta_{2,-1}^{1,0,0,0}$	d_α	$1.43 \cdot 10^{4}$	$2.55 \cdot 10^{4}$					
6	$\Theta_{3,-1}^{2,0,0,0}$	d	$2.85 \cdot 10^{6}$	$4.36 \cdot 10^{6}$					
7	$\Theta_{1,-1}^{0,0,0,0}$	$d_{\alpha\alpha}$	$1.20 \cdot 10^{2}$	$1.51 \cdot 10^{2}$					
8	$\Theta_{3,0}^{0,1,0,0}$	D	$6.33 \cdot 10^{10}$	$1.90 \cdot 10^{10}$	$1.26 \cdot 10^{11}$	$5.48 \cdot 10^{8}$	$9.67 \cdot 10^{10}$	$4.41 \cdot 10^{10}$	$1.33 \cdot 10^{11}$
9	$\Theta_{3,1}^{1,0,0,0}$	D	$1.62 \cdot 10^{10}$	$1.40 \cdot 10^{10}$	$2.47 \cdot 10^{11}$	$7.28 \cdot 10^{8}$	$2.94 \cdot 10^{11}$	$2.51 \cdot 10^{11}$	$1.57 \cdot 10^{11}$
10	$\Theta_{4,0}^{2,0,0,0}$	D	$3.42 \cdot 10^{10}$	$2.66 \cdot 10^{10}$	$1.25 \cdot 10^{11}$	$4.22 \cdot 10^{8}$	$7.30 \cdot 10^{10}$	$8.35 \cdot 10^{10}$	$1.92 \cdot 10^{11}$
11	$\Theta_{2,2}^{0,0,0,0}$	D	$5.58 \cdot 10^{9}$	$1.47 \cdot 10^{10}$	$1.34 \cdot 10^{12}$	$1.56 \cdot 10^{9}$	$1.11 \cdot 10^{12}$	$2.47 \cdot 10^{11}$	$2.94 \cdot 10^{11}$

5.3.3.4 3 derivatives in Q: Linear terms

Num	Kernel	Acts	T_1	T_2	$T_{3,1}$	$T_{3,2}$	$T_{3,3}$	$T_{4,1}$	$T_{4,2}$
1	$\Theta_{1,-1}^{0,0,0,0}$	d	$1.99 \cdot 10^{4}$	$3.22 \cdot 10^{4}$					
2	$\Theta_{2,0}^{0,0,0,0}$	D	$3.51 \cdot 10^{8}$	$7.87 \cdot 10^{7}$	$8.43 \cdot 10^{8}$	$2.95 \cdot 10^{6}$	$4.38 \cdot 10^{8}$	$3.49 \cdot 10^{8}$	$1.31 \cdot 10^{9}$

1	$\Theta_{2,0}^{0,0,0,0}$	D_α	$3.51 \cdot 10^{8}$	$7.87 \cdot 10^{7}$	$8.43 \cdot 10^{8}$	$2.95 \cdot 10^{6}$	$4.38 \cdot 10^{8}$	$3.49 \cdot 10^{8}$	$1.31 \cdot 10^{9}$
2	$\Theta_{1,-1}^{0,0,0,0}$	d_α	$1.99 \cdot 10^{4}$	$3.22 \cdot 10^{4}$					
3	$\Theta_{2,1}^{0,0,0,0}$	D	$9.63 \cdot 10^{9}$	$8.59 \cdot 10^{9}$	$2.45 \cdot 10^{11}$	$8.43 \cdot 10^{8}$	$3.05 \cdot 10^{11}$	$1.40 \cdot 10^{11}$	$5.62 \cdot 10^{10}$
4	$\Theta_{3,0}^{1,0,0,0}$	D	$8.01 \cdot 10^{10}$	$2.18 \cdot 10^{10}$	$1.40 \cdot 10^{11}$	$3.49 \cdot 10^{8}$	$5.62 \cdot 10^{10}$	$1.13 \cdot 10^{11}$	$4.46 \cdot 10^{11}$
5	$\Theta_{2,-1}^{1,0,0,0}$	d	$2.52 \cdot 10^{6}$	$5.12 \cdot 10^{6}$					

5.3.3.5 4 derivatives in Q: Linear terms

Num	Kernel	Acts	T_1	T_2	$T_{3,1}$	$T_{3,2}$	$T_{3,3}$	$T_{4,1}$	$T_{4,2}$
1	$\Theta_{1,-1}^{0,0,0,0}$	d	$1.22 \cdot 10^{7}$	$2.66 \cdot 10^{7}$					
2	$\Theta_{2,0}^{0,0,0,0}$	D	$3.30 \cdot 10^{11}$	$5.80 \cdot 10^{10}$	$7.60 \cdot 10^{11}$	$2.45 \cdot 10^{9}$	$3.98 \cdot 10^{11}$	$4.13 \cdot 10^{11}$	$1.14 \cdot 10^{12}$

5.3.3.6 0 derivatives in Q: Totals

Num	Kernel	D	D_α	$D_{\alpha\alpha}$	$\partial_\alpha^3 D$	$\partial_\alpha^4 D$	d	d_α	$d_{\alpha\alpha}$	$\partial_\alpha^3 d$	$\partial_\alpha^4 d$
1	$\Theta_{1,-1}^{0,0,0,0}$						$3.16 \cdot 10^0$				
2	$\Theta_{2,0}^{0,0,0,0}$	$2.64 \cdot 10^4$	$2.69 \cdot 10^1$								
Total		$2.64 \cdot 10^4$	$2.7 \cdot 10^1$				$3.16 \cdot 10^0$				
1	$\Theta_{2,0}^{0,0,0,0}$		$2.64 \cdot 10^4$	$2.69 \cdot 10^1$							
2	$\Theta_{1,-1}^{0,0,0,0}$							$3.16 \cdot 10^0$			
3	$\Theta_{2,1}^{0,0,0,0}$	$6.46 \cdot 10^6$	$6.08 \cdot 10^3$								
4	$\Theta_{3,0}^{1,0,0,0}$	$6.77 \cdot 10^6$	$3.60 \cdot 10^3$								
5	$\Theta_{2,-1}^{1,0,0,0}$						$5.97 \cdot 10^1$				
Total		$1.32 \cdot 10^7$	$3.61 \cdot 10^4$	$2.69 \cdot 10^1$			$5.97 \cdot 10^1$	$3.16 \cdot 10^0$			
1	$\Theta_{3,0}^{1,0,0,0}$		$6.77 \cdot 10^6$	$3.60 \cdot 10^3$							
2	$\Theta_{2,0}^{0,0,0,0}$			$2.64 \cdot 10^4$	$2.69 \cdot 10^1$						
3	$\Theta_{2,1}^{0,0,0,0}$		$6.46 \cdot 10^6$	$6.08 \cdot 10^3$							
4	$\Theta_{2,-1}^{0,1,0,0}$						$5.47 \cdot 10^5$				
5	$\Theta_{2,-1}^{1,0,0,0}$							$5.97 \cdot 10^1$			
6	$\Theta_{3,-1}^{2,0,0,0}$						$1.05 \cdot 10^4$				
7	$\Theta_{1,-1}^{0,0,0,0}$								$3.16 \cdot 10^0$		
8	$\Theta_{3,0}^{0,1,0,0}$	$7.13 \cdot 10^8$	$7.92 \cdot 10^5$								
9	$\Theta_{3,1}^{1,0,0,0}$	$1.38 \cdot 10^9$	$1.05 \cdot 10^6$								
10	$\Theta_{4,0}^{2,0,0,0}$	$7.57 \cdot 10^8$	$5.70 \cdot 10^5$								
11	$\Theta_{2,2}^{0,0,0,0}$	$4.25 \cdot 10^9$	$2.15 \cdot 10^6$								
Total		$7.11 \cdot 10^9$	$1.78 \cdot 10^7$	$3.61 \cdot 10^4$	$2.69 \cdot 10^1$		$5.58 \cdot 10^5$	$5.97 \cdot 10^1$	$3.16 \cdot 10^0$		
1	$\Theta_{3,0}^{0,1,0,0}$		$7.13 \cdot 10^8$	$7.92 \cdot 10^5$							
2	$\Theta_{3,-1}^{1,1,0,0}$						$1.99 \cdot 10^6$				
3	$\Theta_{3,0}^{1,0,0,0}$			$6.77 \cdot 10^6$	$3.60 \cdot 10^3$						
4	$\Theta_{3,1}^{1,0,0,0}$		$1.38 \cdot 10^9$	$1.05 \cdot 10^6$							
5	$\Theta_{4,0}^{2,0,0,0}$		$7.57 \cdot 10^8$	$5.70 \cdot 10^5$							
6	$\Theta_{2,0}^{0,0,0,0}$				$2.64 \cdot 10^4$	$2.69 \cdot 10^1$					
7	$\Theta_{2,1}^{0,0,0,0}$			$6.46 \cdot 10^6$	$6.08 \cdot 10^3$						

8	$\Theta_{2,2}^{0,0,0,0}$		$4.25 \cdot 10^{9}$	$2.15 \cdot 10^{6}$							
9	$\Theta_{2,-1}^{0,0,1,0}$						$8.43 \cdot 10^{6}$				
10	$\Theta_{2,-1}^{0,1,0,0}$							$5.47 \cdot 10^{5}$			
11	$\Theta_{4,0}^{1,1,0,0}$	$1.24 \cdot 10^{11}$	$1.15 \cdot 10^{8}$								
12	$\Theta_{2,-1}^{1,0,0,0}$								$5.97 \cdot 10^{1}$		
13	$\Theta_{3,-1}^{2,0,0,0}$							$1.05 \cdot 10^{4}$			
14	$\Theta_{4,-1}^{3,0,0,0}$						$1.83 \cdot 10^{6}$				
15	$\Theta_{1,-1}^{0,0,0,0}$									$3.16 \cdot 10^{0}$	
16	$\Theta_{3,0}^{0,0,1,0}$	$8.51 \cdot 10^{11}$	$4.08 \cdot 10^{8}$								
17	$\Theta_{3,1}^{0,1,0,0}$	$1.55 \cdot 10^{11}$	$1.83 \cdot 10^{8}$								
18	$\Theta_{3,2}^{1,0,0,0}$	$8.50 \cdot 10^{11}$	$3.35 \cdot 10^{8}$								
19	$\Theta_{4,1}^{2,0,0,0}$	$1.92 \cdot 10^{11}$	$1.81 \cdot 10^{8}$								
20	$\Theta_{5,0}^{3,0,0,0}$	$1.52 \cdot 10^{11}$	$9.40 \cdot 10^{7}$								
21	$\Theta_{2,3}^{0,0,0,0}$	$4.04 \cdot 10^{12}$	$1.95 \cdot 10^{9}$								
Total		$6.36 \cdot 10^{12}$	$1.04 \cdot 10^{10}$	$1.78 \cdot 10^{7}$	$3.61 \cdot 10^{4}$	$2.69 \cdot 10^{1}$	$1.22 \cdot 10^{7}$	$5.58 \cdot 10^{5}$	$5.97 \cdot 10^{1}$	$3.16 \cdot 10^{0}$	
1	$\Theta_{4,0}^{1,1,0,0}$		$1.24 \cdot 10^{11}$	$1.15 \cdot 10^{8}$							
2	$\Theta_{3,0}^{0,0,1,0}$		$8.51 \cdot 10^{11}$	$4.08 \cdot 10^{8}$							
3	$\Theta_{3,-1}^{1,0,1,0}$						$1.79 \cdot 10^{10}$				
4	$\Theta_{3,0}^{0,1,0,0}$			$7.13 \cdot 10^{8}$	$7.92 \cdot 10^{5}$						
5	$\Theta_{3,1}^{0,1,0,0}$		$1.55 \cdot 10^{11}$	$1.83 \cdot 10^{8}$							
6	$\Theta_{3,-1}^{1,1,0,0}$							$1.99 \cdot 10^{6}$			
7	$\Theta_{4,-1}^{2,1,0,0}$						$3.37 \cdot 10^{8}$				
8	$\Theta_{3,0}^{1,0,0,0}$				$6.77 \cdot 10^{6}$	$3.60 \cdot 10^{3}$					
9	$\Theta_{3,1}^{1,0,0,0}$			$1.38 \cdot 10^{9}$	$1.05 \cdot 10^{6}$						
10	$\Theta_{3,2}^{1,0,0,0}$		$8.50 \cdot 10^{11}$	$3.35 \cdot 10^{8}$							
11	$\Theta_{4,0}^{2,0,0,0}$			$7.57 \cdot 10^{8}$	$5.70 \cdot 10^{5}$						
12	$\Theta_{4,1}^{2,0,0,0}$		$1.92 \cdot 10^{11}$	$1.81 \cdot 10^{8}$							
13	$\Theta_{5,0}^{3,0,0,0}$		$1.52 \cdot 10^{11}$	$9.40 \cdot 10^{7}$							
14	Estimated using an extra cancelation										
15	$\Theta_{2,1}^{0,0,0,0}$				$6.46 \cdot 10^{6}$	$6.08 \cdot 10^{3}$					

16	$\Theta_{2,2}^{0,0,0,0}$			$4.25 \cdot 10^{9}$	$2.15 \cdot 10^{6}$						
17	$\Theta_{2,3}^{0,0,0,0}$		$4.04 \cdot 10^{12}$	$1.95 \cdot 10^{9}$							
18	$\Theta_{2,-1}^{0,0,0,1}$						$6.06 \cdot 10^{9}$				
19	$\Theta_{2,-1}^{0,0,1,0}$							$8.43 \cdot 10^{6}$			
20	$\Theta_{4,0}^{1,0,1,0}$	$1.19 \cdot 10^{14}$	$6.88 \cdot 10^{10}$								
21	$\Theta_{2,-1}^{0,1,0,0}$								$5.47 \cdot 10^{5}$		
22	$\Theta_{4,1}^{1,1,0,0}$	$3.18 \cdot 10^{13}$	$3.13 \cdot 10^{10}$								
23	$\Theta_{5,0}^{2,1,0,0}$	$4.90 \cdot 10^{13}$	$1.81 \cdot 10^{10}$								
24	$\Theta_{2,-1}^{1,0,0,0}$									$5.97 \cdot 10^{1}$	
25	$\Theta_{3,-1}^{0,2,0,0}$						$4.72 \cdot 10^{8}$				
26	$\Theta_{3,-1}^{2,0,0,0}$								$1.05 \cdot 10^{4}$		
27	$\Theta_{4,-1}^{3,0,0,0}$							$1.83 \cdot 10^{6}$			
28	$\Theta_{5,-1}^{4,0,0,0}$						$3.18 \cdot 10^{8}$				
29	$\Theta_{1,-1}^{0,0,0,0}$										$3.16 \cdot 10^{0}$
30	$\Theta_{3,0}^{0,0,0,1}$	$8.64 \cdot 10^{14}$	$3.04 \cdot 10^{11}$								
31	$\Theta_{3,1}^{0,0,1,0}$	$1.67 \cdot 10^{14}$	$1.38 \cdot 10^{11}$								
32	$\Theta_{3,2}^{0,1,0,0}$	$1.13 \cdot 10^{14}$	$6.88 \cdot 10^{10}$								
33	$\Theta_{3,3}^{1,0,0,0}$	$7.85 \cdot 10^{14}$	$3.35 \cdot 10^{11}$								
34	$\Theta_{4,0}^{0,2,0,0}$	$4.64 \cdot 10^{13}$	$2.50 \cdot 10^{10}$								
35	$\Theta_{4,2}^{2,0,0,0}$	$1.25 \cdot 10^{14}$	$5.38 \cdot 10^{10}$								
36	$\Theta_{5,1}^{3,0,0,0}$	$3.32 \cdot 10^{13}$	$3.12 \cdot 10^{10}$								
37	$\Theta_{6,0}^{4,0,0,0}$	$2.80 \cdot 10^{13}$	$1.58 \cdot 10^{10}$								
38	Estimated using an extra cancelation										
Total		$2.36 \cdot 10^{15}$	$7.45 \cdot 10^{12}$	$1.04 \cdot 10^{10}$	$1.78 \cdot 10^{17}$	$9.68 \cdot 10^{3}$	$2.51 \cdot 10^{10}$	$1.22 \cdot 10^{7}$	$5.58 \cdot 10^{5}$	$5.97 \cdot 10^{1}$	$3.16 \cdot 10^{0}$

5.3.3.7 1 derivative in Q: Totals

Num	Kernel	D	D_α	$D_{\alpha\alpha}$	$\partial_\alpha^3 D$	$\partial_\alpha^4 D$	d	d_α	$d_{\alpha\alpha}$	$\partial_\alpha^3 d$	$\partial_\alpha^4 d$
1	$\Theta_{1,-1}^{0,0,0,0}$						$8.55 \cdot 10^{0}$				
2	$\Theta_{2,0}^{0,0,0,0}$	$9.02 \cdot 10^{4}$	$1.03 \cdot 10^{2}$								

Total		$9.02 \cdot 10^4$	$1.03 \cdot 10^2$				$8.55 \cdot 10^0$				
1	$\Theta^{0,0,0,0}_{2,0}$		$9.02 \cdot 10^4$	$1.03 \cdot 10^2$							
2	$\Theta^{0,0,0,0}_{1,-1}$							$8.55 \cdot 10^0$			
3	$\Theta^{0,0,0,0}_{2,1}$	$2.17 \cdot 10^7$	$2.25 \cdot 10^4$								
4	$\Theta^{1,0,0,0}_{3,0}$	$2.33 \cdot 10^7$	$9.53 \cdot 10^3$								
5	$\Theta^{1,0,0,0}_{2,-1}$						$2.43 \cdot 10^2$				
Total		$4.50 \cdot 10^7$	$1.22 \cdot 10^5$	$1.03 \cdot 10^2$			$2.43 \cdot 10^2$	$8.55 \cdot 10^0$			
1	$\Theta^{1,0,0,0}_{3,0}$		$2.33 \cdot 10^7$	$9.53 \cdot 10^3$							
2	$\Theta^{0,0,0,0}_{2,0}$			$9.02 \cdot 10^4$	$1.03 \cdot 10^2$						
3	$\Theta^{0,0,0,0}_{2,1}$		$2.17 \cdot 10^7$	$2.25 \cdot 10^4$							
4	$\Theta^{0,1,0,0}_{2,-1}$						$2.19 \cdot 10^6 2$				
5	$\Theta^{1,0,0,0}_{2,-1}$							$2.43 \cdot 10^2$			
6	$\Theta^{2,0,0,0}_{3,-1}$						$3.99 \cdot 10^4$				
7	$\Theta^{0,0,0,0}_{1,-1}$								$8.55 \cdot 10^0$		
8	$\Theta^{0,1,0,0}_{3,0}$	$2.41 \cdot 10^9$	$1.75 \cdot 10^6$								
9	$\Theta^{1,0,0,0}_{3,1}$	$4.53 \cdot 10^9$	$3.80 \cdot 10^6$								
10	$\Theta^{2,0,0,0}_{4,0}$	$2.62 \cdot 10^9$	$1.63 \cdot 10^6$								
11	$\Theta^{0,0,0,0}_{2,2}$	$1.47 \cdot 10^{10}$	$7.01 \cdot 10^6$								
Total		$2.43 \cdot 10^{10}$	$5.92 \cdot 10^7$	$1.22 \cdot 10^5$	$1.03 \cdot 10^2$		$2.23 \cdot 10^6$	$2.43 \cdot 10^2$	$8.55 \cdot 10^0$		
1	$\Theta^{0,1,0,0}_{3,0}$		$2.41 \cdot 10^9$	$1.75 \cdot 10^6$							
2	$\Theta^{1,1,0,0}_{3,-1}$						$7.87 \cdot 10^6$				
3	$\Theta^{1,0,0,0}_{3,0}$			$2.33 \cdot 10^7$	$9.53 \cdot 10^3$						
4	$\Theta^{1,0,0,0}_{3,1}$	D_α	$4.53 \cdot 10^9$	$3.80 \cdot 10^6$							
5	$\Theta^{2,0,0,0}_{4,0}$	D_α	$2.62 \cdot 10^9$	$1.63 \cdot 10^6$							
6	$\Theta^{0,0,0,0}_{2,0}$	$\partial^3_\alpha D$	$D_{\alpha\alpha}$	D_α	$9.02 \cdot 10^4$	$1.03 \cdot 10^2$					
7	$\Theta^{0,0,0,0}_{2,1}$	$D_{\alpha\alpha}$	D_α	$2.17 \cdot 10^7$	$2.25 \cdot 10^4$						
8	$\Theta^{0,0,0,0}_{2,2}$	D_α	$1.47 \cdot 10^{10}$	$7.01 \cdot 10^6$							
9	$\Theta^{0,0,1,0}_{2,-1}$						$3.01 \cdot 10^7$				
10	$\Theta^{0,1,0,0}_{2,-1}$	d_α						$2.19 \cdot 10^6$			
11	$\Theta^{1,1,0,0}_{4,0}$	$4.49 \cdot 10^{11}$	$2.91 \cdot 10^8$								

12	$\Theta^{1,0,0,0}_{2,-1}$	$d_{\alpha\alpha}$	d_α						$2.43 \cdot 10^2$		
13	$\Theta^{2,0,0,0}_{3,-1}$	d_α						$3.99 \cdot 10^4$			
14	$\Theta^{3,0,0,0}_{4,-1}$						$6.77 \cdot 10^6$				
15	$\Theta^{0,0,0,0}_{1,-1}$	$\partial^3_\alpha d$	$d_{\alpha\alpha}$	d_α						$8.55 \cdot 10^0$	
16	$\Theta^{0,0,1,0}_{3,0}$	$3.04 \cdot 10^{12}$	$1.23 \cdot 10^9$								
17	$\Theta^{0,1,0,0}_{3,1}$	$5.30 \cdot 10^{11}$	$6.99 \cdot 10^8$								
18	$\Theta^{1,0,0,0}_{3,2}$	$2.89 \cdot 10^{12}$	$1.04 \cdot 10^9$								
19	$\Theta^{2,0,0,0}_{4,1}$	$6.24 \cdot 10^{11}$	$6.44 \cdot 10^8$								
20	$\Theta^{3,0,0,0}_{5,0}$	$5.40 \cdot 10^{11}$	$2.79 \cdot 10^8$								
21	$\Theta^{0,0,0,0}_{2,3}$	$1.43 \cdot 10^{13}$	$6.69 \cdot 10^9$								
Total		$2.24 \cdot 10^{13}$	$3.51 \cdot 10^{10}$	$5.92 \cdot 10^7$	$1.22 \cdot 10^5$	$1.03 \cdot 10^2$	$4.48 \cdot 10^7$	$2.23 \cdot 10^6$	$2.43 \cdot 10^2$	$8.55 \cdot 10^0$	

5.3.3.8 2 derivatives in Q: Totals

Num	Kernel	D	D_α	$D_{\alpha\alpha}$	$\partial^3_\alpha D$	$\partial^4_\alpha D$	d	d_α	$d_{\alpha\alpha}$	$\partial^3_\alpha d$	$\partial^4_\alpha d$
1	$\Theta^{0,0,0,0}_{1,-1}$						$2.53 \cdot 10^2$				
2	$\Theta^{0,0,0,0}_{2,0}$	$1.86 \cdot 10^7$	$1.71 \cdot 10^4$								
Total		$1.86 \cdot 10^7$	$1.71 \cdot 10^4$				$2.53 \cdot 10^2$				
1	$\Theta^{0,0,0,0}_{2,0}$		$1.86 \cdot 10^7$	$1.71 \cdot 10^4$							
2	$\Theta^{0,0,0,0}_{1,-1}$							$2.53 \cdot 10^2$			
3	$\Theta^{0,0,0,0}_{2,1}$	$4.54 \cdot 10^9$	$4.22 \cdot 10^6$								
4	$\Theta^{1,0,0,0}_{3,0}$	$4.79 \cdot 10^9$	$2.62 \cdot 10^6$								
5	$\Theta^{1,0,0,0}_{2,-1}$						$3.98 \cdot 10^4$				
Total		$9.33 \cdot 10^9$	$2.54 \cdot 10^7$	$1.71 \cdot 10^4$			$3.98 \cdot 10^4$	$2.53 \cdot 10^2$			
1	$\Theta^{1,0,0,0}_{3,0}$		$4.79 \cdot 10^9$	$2.62 \cdot 10^6$							
2	$\Theta^{0,0,0,0}_{2,0}$			$1.86 \cdot 10^7$	$1.71 \cdot 10^4$						
3	$\Theta^{0,0,0,0}_{2,1}$		$4.54 \cdot 10^9$	$4.22 \cdot 10^6$							
4	$\Theta^{0,1,0,0}_{2,-1}$						$3.24 \cdot 10^8$				
5	$\Theta^{1,0,0,0}_{2,-1}$							$3.98 \cdot 10^4$			
6	$\Theta^{2,0,0,0}_{3,-1}$						$7.21 \cdot 10^6$				

7	$\Theta^{0,0,0,0}_{1,-1}$								$2.53 \cdot 10^{2}$		
8	$\Theta^{0,1,0,0}_{3,0}$	$4.82 \cdot 10^{11}$	$5.48 \cdot 10^{8}$								
9	$\Theta^{1,0,0,0}_{3,1}$	$9.79 \cdot 10^{11}$	$7.28 \cdot 10^{8}$								
10	$\Theta^{2,0,0,0}_{4,0}$	$5.35 \cdot 10^{11}$	$4.22 \cdot 10^{8}$								
11	$\Theta^{0,0,0,0}_{2,2}$	$3.01 \cdot 10^{12}$	$1.56 \cdot 10^{9}$								
Total		$5.01 \cdot 10^{12}$	$1.26 \cdot 10^{10}$	$2.54 \cdot 10^{7}$	$1.71 \cdot 10^{4}$		$3.31 \cdot 10^{8}$	$3.98 \cdot 10^{4}$	$2.53 \cdot 10^{2}$		

5.3.3.9 3 derivatives in Q: Totals

Num	Kernel	D	D_α	$D_{\alpha\alpha}$	$\partial^3_\alpha D$	$\partial^4_\alpha D$	d	d_α	$d_{\alpha\alpha}$	$\partial^3_\alpha d$	$\partial^4_\alpha d$
1	$\Theta^{0,0,0,0}_{1,-1}$						$5.22 \cdot 10^{4}$				
2	$\Theta^{0,0,0,0}_{2,0}$	$3.37 \cdot 10^{9}$	$2.95 \cdot 10^{6}$								
Total		$3.37 \cdot 10^{9}$	$2.95 \cdot 10^{6}$				$5.22 \cdot 10^{4}$				
1	$\Theta^{0,0,0,0}_{2,0}$		$3.37 \cdot 10^{9}$	$2.95 \cdot 10^{6}$							
2	$\Theta^{0,0,0,0}_{1,-1}$							$5.22 \cdot 10^{4}$			
3	$\Theta^{0,0,0,0}_{2,1}$	$7.65 \cdot 10^{11}$	$8.43 \cdot 10^{8}$								
4	$\Theta^{1,0,0,0}_{3,0}$	$8.57 \cdot 10^{11}$	$3.49 \cdot 10^{8}$								
5	$\Theta^{1,0,0,0}_{2,-1}$						$7.64 \cdot 10^{6}$				
Total		$1.62 \cdot 10^{12}$	$4.56 \cdot 10^{9}$	$2.95 \cdot 10^{6}$			$7.64 \cdot 10^{6}$	$5.22 \cdot 10^{4}$			

5.3.3.10 4 derivatives in Q: Totals

Num	Kernel	D	D_α	$D_{\alpha\alpha}$	$\partial^3_\alpha D$	$\partial^4_\alpha D$	d	d_α	$d_{\alpha\alpha}$	$\partial^3_\alpha d$	$\partial^4_\alpha d$
1	$\Theta^{0,0,0,0}_{1,-1}$						$3.88 \cdot 10^{7}$				
2	$\Theta^{0,0,0,0}_{2,0}$	$3.10 \cdot 10^{12}$	$2.45 \cdot 10^{9}$								
Total		$3.10 \cdot 10^{12}$	$2.45 \cdot 10^{9}$				$3.88 \cdot 10^{7}$				

5.3.3.11 Writing the linear system for D and its derivatives

We end this section packing all the previous estimates and getting the final estimates for the linear part of the energy concerning D and its derivatives. The estimates coming from the previous tables are summarized in

$$\begin{pmatrix} 2.64\cdot 10^4 & 2.7\cdot 10^1 & 0 & 0 & 0 \\ 1.33\cdot 10^7 & 3.62\cdot 10^4 & 2.69\cdot 10^1 & 0 & 0 \\ 7.22\cdot 10^9 & 1.81\cdot 10^7 & 3.63\cdot 10^4 & 2.69\cdot 10^1 & 0 \\ 6.47\cdot 10^{12} & 1.06\cdot 10^{10} & 1.82\cdot 10^7 & 3.64\cdot 10^4 & 2.69\cdot 10^1 \\ 2.49\cdot 10^{15} & 7.69\cdot 10^{12} & 1.08\cdot 10^{10} & 1.84\cdot 10^7 & 1.01\cdot 10^4 \end{pmatrix}$$

We are left thus with the estimates coming from the term $Q^2(z)BR(z,\omega)-Q^2(x)BR(x,\gamma))$, which at a linear order are of the form

$$BR(x,\gamma)\underbrace{\frac{1}{8}\left\langle \frac{1+x^4}{x}, \overline{3x^2-\frac{1}{x^2}}\right\rangle}_{F} D + O(D^2).$$

We show now the different L^∞ estimates depending on the number of derivatives we are taking in every term

In BR \In F	0	1	2	3	4
0	$1.77\cdot 10^2$	$1.22\cdot 10^3$	$1.02\cdot 10^5$	$2.14\cdot 10^7$	$1.56\cdot 10^{10}$
1	$1.42\cdot 10^4$	$5.75\cdot 10^4$	$1.35\cdot 10^7$	$2.05\cdot 10^9$	–
2	$7.97\cdot 10^6$	$3.71\cdot 10^7$	$7.32\cdot 10^9$	–	–
3	$5.54\cdot 10^9$	$2.44\cdot 10^{10}$	–	–	–
4	$6.40\cdot 10^{12}$	–	–	–	–

This means the following estimates for the derivatives:

Derivatives in (BRF)	0	1	2	3	4
Estimates	$1.77\cdot 10^2$	$1.54\cdot 10^4$	$8.18\cdot 10^6$	$5.71\cdot 10^9$	$6.56\cdot 10^{12}$

We summarize the contribution of these terms to the derivative of $E(t)$:

$$\begin{pmatrix} 1.77\cdot 10^2 & 0 & 0 & 0 & 0 \\ 1.54\cdot 10^4 & 1.77\cdot 10^2 & 0 & 0 & 0 \\ 8.18\cdot 10^6 & 3.08\cdot 10^4 & 1.77\cdot 10^2 & 0 & 0 \\ 5.71\cdot 10^9 & 2.45\cdot 10^7 & 4.62\cdot 10^4 & 1.77\cdot 10^2 & 0 \\ 6.56\cdot 10^{12} & 2.28\cdot 10^{10} & 4.91\cdot 10^7 & 6.16\cdot 10^4 & 1.77\cdot 10^2 \end{pmatrix}\begin{pmatrix} \|D\|_{L^2} \\ \|\partial_\alpha D\|_{L^2} \\ \|\partial_\alpha^2 D\|_{L^2} \\ \|\partial_\alpha^3 D\|_{L^2} \\ \|\partial_\alpha^4 D\|_{L^2} \end{pmatrix}$$

The total contribution is thus given by

$$\begin{pmatrix} 2.66\cdot 10^4 & 2.7\cdot 10^1 & 0 & 0 & 0 \\ 1.33\cdot 10^7 & 3.64\cdot 10^4 & 2.69\cdot 10^1 & 0 & 0 \\ 7.22\cdot 10^9 & 1.81\cdot 10^7 & 3.65\cdot 10^4 & 2.69\cdot 10^1 & 0 \\ 6.47\cdot 10^{12} & 1.06\cdot 10^{10} & 1.82\cdot 10^7 & 3.66\cdot 10^4 & 2.69\cdot 10^1 \\ 2.49\cdot 10^{15} & 7.71\cdot 10^{12} & 1.08\cdot 10^{10} & 1.85\cdot 10^7 & 1.03\cdot 10^4 \end{pmatrix} \begin{pmatrix} \|D\|_{L^2} \\ \|\partial_\alpha D\|_{L^2} \\ \|\partial_\alpha^2 D\|_{L^2} \\ \|\partial_\alpha^3 D\|_{L^2} \\ \|\partial_\alpha^4 D\|_{L^2} \end{pmatrix}$$

Finally, one can compute the largest eigenvalue λ_1 of the matrix. We obtain that it equals $7.94\cdot 10^4$. Since the growth of the energy is given roughly by $\exp(\lambda_1 T_g)/\lambda_1\|f\|$ an this has to be smaller than $\min \partial_\alpha x^1(\alpha, T_g)$, which in our case is approximately 0.1. The estimates prove to be insufficient, although promising, since just an order of magnitude less could give us enough room to prove the theorem (we assume the norms of f can be bounded by a constant of the order of 10^{-6}). We expect to lower λ_1 in the near future. Some ideas about how this can be carried out are presented in the next section.

5.3.4 Future improvements

We give here a list of possible optimizations that although at this moment untested, we believe may lead us to get better estimates. The first one concerns the splitting of the kernels for the computation of the constant $C(t)$. An improvement of the splitting may consist in changing the approximation of the Kernel. Previously it was done by Cauchy transforms over straight lines (tangent at the curve). We propose to approximate it by Cauchy transforms over other curves. The most simple example is circumferences. In this case, let $r^\alpha(\beta)$ be the osculating circle at $x(\alpha)$, parameterized by β. Thus, we can approximate the kernel

$$\Theta_{2,0}^{0,0,0,0} = \frac{\gamma(\beta)}{x_\beta^2(\beta)}\left(\frac{x_\beta^2(\beta)}{(x(\alpha)-x(\beta))^2} - \frac{(r_\beta^\alpha(\beta))^2}{(r^\alpha(\alpha)-r^\alpha(\beta))^2} - c(\alpha)\frac{r_\beta^\alpha(\beta)}{r^\alpha(\alpha)-r^\alpha(\beta)}\right),$$

where

$$c(\alpha) = \frac{r_{\beta\beta}^\alpha(\alpha)x_\beta(\alpha) - r_{\beta\beta}^\alpha(\alpha)x_\beta(\alpha)}{r_\beta^\alpha(\alpha)x_\beta(\alpha)}.$$

For the case of the circle, it is easy to see that

$$\frac{r_\beta^\alpha(\beta)}{r^\alpha(\alpha)-r^\alpha(\beta)} = \frac{e^{i\alpha}}{2}\left(1+\frac{i}{\tan\left(\frac{\beta-\alpha}{2}\right)}\right)$$
$$\frac{(r_\beta^\alpha(\beta))^2}{(r^\alpha(\alpha)-r^\alpha(\beta))^2} = \frac{e^{2i\alpha}}{4}\left(2+\frac{2i}{\tan\left(\frac{\beta-\alpha}{2}\right)} - \frac{1}{\sin^2\left(\frac{\beta-\alpha}{2}\right)}\right),$$

which allows us to write the previous operator as the usual Hilbert Transform plus other terms which are harmless and easy to estimate.

The drawback of this method is that flat regions (with a very small curvature) might lead to high numerical errors since the curvature is infinite at those points. Perhaps a mixed method (integrating in some regions over straight lines and in other over circumferences) might yield better results. Another option to try is to consider a family curves which have a circular part (an arc of a circle) and are prolonged by straight segments, taking the best element of the family to split the kernel.

Other improvements can consist on a different representation of the function: instead of representing it by splines in space-time, one could try to represent it by wavelets or by Chebychev polynomials in space and splines in time. The objective is that a big enough finite dimensional subspace of these function spaces is mapped by the Birkhoff-Rott integral into itself, making the orthogonal projection over this subspace almost zero. This could lead us to a better control of the norm of the orthogonal projection in time.

Finally, one could try to perform a higher order non-rigorous simulation in order to get a better approximation of the real solution. By higher order we mean an improvement in the following senses: multiple precision for the double representation and higher order approximation of the integral approximations. We believe a higher order scheme in time will not produce significantly better results. Of course, there is a tradeoff between the computation time and the precision we can get. Therefore, in order to achieve these results the code needs to be highly optimized and perhaps run in parallel.

5.4 Proof of Theorem 5.1.2

In this section, we will prove the stability Theorem 5.1.2.

The equations are:

$$\text{SPLASH}\left\{\begin{array}{ll} z_t & = Q^2_z BR + cz_\alpha \\ c & = \dfrac{\alpha+\pi}{2\pi}\displaystyle\int_{-\pi}^{\pi}(Q^2BR)_\alpha\frac{z_\alpha}{|z_\alpha|^2}d\alpha - \int_{-\pi}^{\alpha}(Q^2BR)_\beta\frac{z_\beta}{|z_\beta|^2}d\beta \\ \omega_t & +2BR_t\cdot z_\alpha = -(Q^2)_\alpha|BR|^2 + 2cBR_\alpha\cdot z_\alpha + (c\varpi)_\alpha \\ & -\left(\dfrac{Q^2\varpi^2}{4|z_\alpha|^2}\right)_\alpha - 2(P_2^{-1}(z))_\alpha \end{array}\right.$$

$$\text{APPROX}\left\{\begin{array}{ll} x_t & = Q^2(x)BR(x,\gamma) + bx_\alpha + f \\ b & = \underbrace{\dfrac{\alpha+\pi}{2\pi}\displaystyle\int_{-\pi}^{\pi}(Q^2BR)_\alpha\frac{x_\alpha}{|x_\alpha|^2}d\alpha - \int_{-\pi}^{\alpha}(Q^2BR)_\beta\frac{x_\alpha}{|x_\alpha|^2}d\beta}_{b_s} \\ & + \underbrace{\dfrac{\alpha+\pi}{2\pi}\displaystyle\int_{-\pi}^{\pi}f_\alpha\frac{x_\alpha}{|x_\alpha|^2}d\alpha - \int_{-\pi}^{\alpha}f_\beta\frac{x_\beta}{|x_\beta|^2}d\beta}_{b_e} \\ \gamma_t & +2BR_t(x,\gamma)\cdot x_\alpha = -(Q^2(x))_\alpha|BR(x,\gamma)|^2 + 2bBR_\alpha(x,\gamma)\cdot x_\alpha + (b\gamma)_\alpha \\ & \qquad -\left(\dfrac{Q^2(x)\gamma^2}{4|x_\alpha|^2}\right)_\alpha - 2(P_2^{-1}(x))_\alpha + g \end{array}\right.$$

where
$$BR(z,\varpi)(\alpha)=\frac{1}{2\pi}PV\int_{-\pi}^{\pi}\frac{(z(\alpha)-z(\alpha-\beta))^{\perp}}{|z(\alpha)-z(\alpha-\beta)|^2}\varpi(\alpha-\beta)d\beta$$
f will be the error for z and g will be the error for ω.

5.4.1 Computing the difference $z-x$ and $\omega-\gamma$

We define now:
$$D\equiv z-x,\quad d\equiv\omega-\gamma,\quad \mathcal{D}\equiv\varphi-\psi$$
The energy
$$E(t)\equiv\frac{1}{2}\left(\|D\|^2_{L^2}+\int_{-\pi}^{\pi}\frac{Q^2_z}{|z_\alpha|^2}\sigma_z|\partial^4_\alpha D|^2+\|d\|^2_{H^2}+\|\mathcal{D}\|^2_{H^{3+\frac{1}{2}}}\right)$$
and the Rayleigh-Taylor condition
$$\sigma_z\equiv\left(BR_t+\frac{\varphi}{|z_\alpha|}BR_\alpha\right)\cdot z^{\perp}_\alpha+\frac{\omega}{2|z_\alpha|^2}\left(z_{\alpha t}+\frac{\varphi}{|z_\alpha|}z_{\alpha\alpha}\right)\cdot z^{\perp}_\alpha$$
$$+Q\left|BR+\frac{\omega}{2|z_\alpha|^2}z_\alpha\right|^2\nabla Q\cdot z^{\perp}_\alpha-(\nabla P_2^{-1})(z)\cdot z^{\perp}_\alpha$$
Note that $\sigma_z>0$. We shall show that
$$\left|\frac{d}{dt}E(t)\right|\le\mathcal{C}(t)(E(t)+E^k(t))+c\delta(t)$$
where
$$\mathcal{C}(t)=\mathcal{C}(\|x\|_{H^{5+\frac{1}{2}}}(t),\|\gamma\|_{H^{3+\frac{1}{2}}}(t),\|\psi\|_{H^{4+\frac{1}{2}}}(t),\|F(x)\|_{L^\infty}(t))$$
and
$$\delta(t)=(\|f\|_{H^{5+\frac{1}{2}}}(t),\|g\|_{H^{3+\frac{1}{2}}}(t))^k,\ k\text{ big enough}$$
depend on the norms of f and g.

Remark 5.4.1 *From now on, we will denote $E(t)+E(t)^k$ by $P(E(t))$.*

$\frac{1}{2}\frac{d}{dt}\|D\|^2_{L^2}\le CP(E(t))+\delta(t)$ is left to the reader. We compute
$$\frac{1}{2}\frac{d}{dt}\int_{-\pi}^{\pi}\frac{Q^2_z}{|z_\alpha|^2}\sigma_z|\partial^4_\alpha D|^2=\frac{1}{2}\int_{-\pi}^{\pi}\frac{(Q^2_z\sigma_z)_t}{|z_\alpha|^2}|\partial^4_\alpha D|^2+\int_{-\pi}^{\pi}\frac{Q^2_z}{|z_\alpha|^2}\sigma_z\partial^4_\alpha D\partial^4_\alpha D_t$$
The first integral is easy to bound by $CP(E(t))$, we proceed as in the local existence theorem 3.5.1. We split
$$I=\int_{-\pi}^{\pi}\frac{Q^2_z}{|z_\alpha|^2}\sigma_z\partial^4_\alpha D\partial^4_\alpha D_t=I_1+I_2+I_3$$

where

$$I_1 = \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2} \sigma_z \partial_\alpha^4 D \partial_\alpha^4 (Q_z^2 BR(z,\omega) - Q_x^2 BR(x,\gamma)) d\alpha$$
$$I_2 = \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2} \sigma_z \partial_\alpha^4 D \partial_\alpha^4 (c z_\alpha - b x_\alpha) d\alpha$$
$$I_3 = \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2} \sigma_z \partial_\alpha^4 D \partial_\alpha^4 f d\alpha$$

We have:

$$I_3 \leq \frac{1}{2} \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2} \sigma_z |\partial_\alpha^4 D|^2 d\alpha + \frac{1}{2} \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2} \sigma_z |\partial_\alpha^4 f|^2 d\alpha \leq CP(E(t)) + \frac{\|Q_z^2 \sigma_z\|_{L^\infty}}{2} \delta(t)$$

Thus, we are done with I_3. We now split

$$I_1 = \text{l.o.t} + I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4}$$
$$I_{1,1} = \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2} \sigma_z \partial_\alpha^4 D (\partial_\alpha^4 (Q_z^2) BR(z,\omega) - \partial_\alpha^4 (Q_x^2) BR(x,\gamma)) d\alpha$$
$$I_{1,2} = \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2} \sigma_z \partial_\alpha^4 D \left(Q_z^2 \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\partial_\alpha^4 z(\alpha) - \partial_\alpha^4 z(\alpha-\beta))^\perp}{|z(\alpha) - z(\alpha-\beta)|^2} \omega(\alpha-\beta) d\beta \right.$$
$$\left. - Q_x^2 \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\partial_\alpha^4 x(\alpha) - \partial_\alpha^4 x(\alpha-\beta))^\perp}{|x(\alpha) - x(\alpha-\beta)|^2} \gamma(\alpha-\beta) d\beta \right) d\alpha$$
$$I_{1,3} = \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2} \sigma_z \partial_\alpha^4 D$$
$$\times \left(Q_z^2 \frac{-1}{\pi} \int_{-\pi}^{\pi} \frac{(z(\alpha) - z(\alpha-\beta))^\perp}{|z(\alpha) - z(\alpha-\beta)|^4} (z(\alpha) - z(\alpha-\beta)) \cdot (\partial_\alpha^4 z(\alpha) - \partial_\alpha^4 z(\alpha-\beta)) \omega(\alpha-\beta) d\beta \right.$$
$$\left. + Q_x^2 \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{(x(\alpha) - x(\alpha-\beta))^\perp}{|x(\alpha) - x(\alpha-\beta)|^4} (x(\alpha) - x(\alpha-\beta)) \cdot (\partial_\alpha^4 x(\alpha) - \partial_\alpha^4 x(\alpha-\beta)) \gamma(\alpha-\beta) d\beta \right)$$
$$I_{1,4} = \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2} \sigma_z \partial_\alpha^4 D \partial_\alpha^4 (Q_z^2 BR(z, \partial_\alpha^4 \omega) - Q_x^2 BR(x, \partial_\alpha^4 \gamma)) d\alpha$$

where l.o.t stands for low order terms, nice terms easier to deal with.

$$I_{1,1} = \text{l.o.t} + I_{1,1,1} \text{ where}$$

$$I_{1,1,1} = 2 \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2} \sigma_z \partial_\alpha^4 D (\nabla Q(z) \cdot \partial_\alpha^4 z BR(z,\omega) - \nabla Q(x) \cdot \partial_\alpha^4 x BR(x,\gamma)) d\alpha$$
$$= 2 \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2} \sigma_z \partial_\alpha^4 D \nabla Q(z) \cdot \partial_\alpha^4 D BR(z,\omega) d\alpha$$
$$+ 2 \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2} \sigma_z \partial_\alpha^4 D (\nabla Q(z) \cdot \partial_\alpha^4 x BR(z,\omega) - \nabla Q(x) \cdot \partial_\alpha^4 x BR(x,\gamma)) d\alpha$$

$$
\begin{aligned}
&\leq 2\int_{-\pi}^{\pi}\frac{Q_z^2}{|z_\alpha|^2}\sigma_z|\partial_\alpha^4 D|^2 d\alpha \underbrace{\|\nabla Q(z)BR(z,\omega)\|_{L^\infty}}_{\text{bounded as for local existence}}\\
&+\int_{-\pi}^{\pi}\frac{Q_z^2}{|z_\alpha|^2}\sigma_z|\partial_\alpha^4 D|^2+\underbrace{\int_{-\pi}^{\pi}\frac{Q_z^2}{|z_\alpha|^2}\sigma_z|\nabla Q(z)\cdot\partial_\alpha^4 xBR(z,\omega)-\nabla Q(x)\cdot\partial_\alpha^4 xBR(x,\gamma)|^2 d\alpha}_{\text{l.o.t in } D \text{ and } d}\\
&\leq CP(E(t))
\end{aligned}
$$

which means $I_{1,1}$ is done.
From now on we will denote

$$\Delta_\beta z(\alpha)=z(\alpha)-z(\alpha-\beta)$$

$I_{1,2}=I_{1,2,1}+I_{1,2,2}+I_{1,2,3}+I_{1,2,4}$ where

$$
\begin{aligned}
I_{1,2,1}&=\int_{-\pi}^{\pi}\frac{Q_z^2}{|z_\alpha|^2}\sigma_z\partial_\alpha^4 DQ_z^2\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\Delta_\beta\partial_\alpha^4 D^\perp(\alpha)}{|\Delta_\beta z(\alpha)|^2}\omega(\alpha-\beta)d\beta d\alpha\\
I_{1,2,2}&=\int_{-\pi}^{\pi}\frac{Q_z^2}{|z_\alpha|^2}\sigma_z\partial_\alpha^4 DQ_z^2\frac{1}{2\pi}\int_{-\pi}^{\pi}\Delta_\beta\partial_\alpha^4 x^\perp(\alpha)\left(\frac{1}{|\Delta_\beta z(\alpha)|^2}-\frac{1}{|\Delta_\beta x(\alpha)|^2}\right)\omega(\alpha-\beta)d\beta d\alpha\\
I_{1,2,3}&=\int_{-\pi}^{\pi}\frac{Q_z^2}{|z_\alpha|^2}\sigma_z\partial_\alpha^4 DQ_z^2\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\Delta_\beta\partial_\alpha^4 x^\perp(\alpha)}{|\Delta_\beta x(\alpha)|^2}d(\alpha-\beta)d\beta d\alpha\\
I_{1,2,4}&=\int_{-\pi}^{\pi}\frac{Q_z^2}{|z_\alpha|^2}\sigma_z\partial_\alpha^4 D(Q_z^2-Q_x^2)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\Delta_\beta\partial_\alpha^4 x^\perp(\alpha)}{|\Delta_\beta x(\alpha)|^2}\gamma(\alpha-\beta)d\beta d\alpha
\end{aligned}
$$

$$
\begin{aligned}
I_{1,2,1}&=\int_{-\pi}^{\pi}\frac{Q_z^4}{|z_\alpha|^2}\sigma_z\partial_\alpha^4 D(\alpha)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\Delta_{\alpha-\beta}\partial_\alpha^4 D^\perp(\alpha)}{|\Delta_{\alpha-\beta}z(\alpha)|^2}\omega(\alpha-\beta)d\beta d\alpha\\
&=\frac{1}{|z_\alpha|^2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\partial_\alpha^4 D\frac{\Delta_{\alpha-\beta}\partial_\alpha^4 D^\perp(\alpha)}{|\Delta_{\alpha-\beta}z(\alpha)|^2}\left(\frac{Q_z^4(\alpha)\sigma_z(\alpha)\omega(\beta)-Q_z^4(\beta)\sigma_z(\beta)\omega(\alpha)}{2}\right.\\
&\left.+\underbrace{\frac{Q_z^4(\alpha)\sigma_z(\alpha)\omega(\beta)+Q_z^4(\beta)\sigma_z(\beta)\omega(\alpha)}{2}}_{\text{this is zero as in local existence } (\partial_\alpha^4 D\cdot\partial_\alpha^4 D^\perp=0)}\right)d\alpha d\beta
\end{aligned}
$$

$$\Rightarrow I_{1,2,1}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial_\alpha^4 D}{|z_\alpha|^2}\int_{-\pi}^{\pi}\underbrace{\frac{\Delta_{\alpha-\beta}\partial_\alpha^4 D^\perp(\alpha)}{|\Delta_{\alpha-\beta}z(\alpha)|^2}}_{\substack{\text{Hilbert transform}\\ \text{applied to } \partial_\alpha^4 D^\perp(\alpha)}}\left(\frac{Q_z^4(\alpha)\sigma_z(\alpha)\omega(\beta)-Q_z^4(\beta)\sigma_z(\beta)\omega(\alpha)}{2}\right)$$

$$\Rightarrow I_{1,2,1}\leq CP(E(t))$$

For $I_{1,2,2}$ we can make a trick to get less derivatives in x.

$$I_{1,2,2} = I^1_{1,2,2} + I^2_{1,2,2} + I^3_{1,2,2}$$

$$I^3_{1,2,2} = \frac{1}{2}\int_{-\pi}^{\pi} \frac{Q_z^4}{|z_\alpha|^2}\sigma_z\partial_\alpha^4 D\omega(\alpha)\left(\frac{1}{|z_\alpha|^2} - \frac{1}{|x_\alpha|^2}\right)\overbrace{\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\Delta_\beta\partial_\alpha^4 x^\perp(\alpha)}{\beta^2}d\beta}^{\Lambda\partial_\alpha^4 x}\, d\alpha$$

$$I^2_{1,2,2} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{Q_z^4}{|z_\alpha|^2}\sigma_z\partial_\alpha^4 D\int_{-\pi}^{\pi}\Delta_\beta\partial_\alpha^4 x^\perp(\alpha)\left(\frac{1}{|\Delta_\beta z(\alpha)|^2} - \frac{1}{|z_\alpha(\alpha)|^2\beta^2} + \overbrace{\frac{z_\alpha\cdot z_{\alpha\alpha}}{|z_\alpha|^4\beta}}^{=0}\right.$$
$$\left. - \left(\frac{1}{|\Delta_\beta x(\alpha)|^2} - \frac{1}{|x_\alpha(\alpha)|^2\beta^2} + \overbrace{\frac{x_\alpha\cdot x_{\alpha\alpha}}{|x_\alpha|^4\beta}}^{=0}\right)\right)\omega(\alpha)d\beta d\alpha$$

$$I^1_{1,2,2} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{Q_z^4}{|z_\alpha|^2}\sigma_z\partial_\alpha^4 D\int_{-\pi}^{\pi}\Delta_\beta\partial_\alpha^4 x^\perp(\alpha)\left(\frac{1}{|\Delta_\beta z(\alpha)|^2} - \frac{1}{|\Delta_\beta x(\alpha)|^2}\right)(\omega(\alpha-\beta)-\omega(\alpha))\,d\beta d\alpha$$

We use that $\left|\frac{1}{|z_\alpha|^2} - \frac{1}{|x_\alpha|^2}\right| \leq \frac{|x_\alpha|+|z_\alpha|}{|z_\alpha|^2|x_\alpha|^2}|D_\alpha|$ to find that

$$I^3_{1,2,2} \leq \frac{1}{4}\int_{-\pi}^{\pi}\frac{Q_z^2}{|z_\alpha|^2}\sigma_z|\partial_\alpha^4 D|^2 + \|Q_z\|^6_{L^\infty}\|\sigma_z\|_{L^\infty}\|\omega\|^2_{L^\infty}\left(\frac{|x_\alpha|+|z_\alpha|}{|z_\alpha|^2|x_\alpha|^2}\right)^2\overbrace{\|D_\alpha\|^2_{L^\infty}}^{\substack{\text{Sobolev}\\ \text{inequalities}}}\ \overbrace{\|\Lambda\partial_\alpha^4 x\|^2_{L^2}}^{\text{Control of }\|x\|_{H^5}}$$
$$\leq CP(E(t))$$

We can use that

$$\left|\left(\frac{1}{|\Delta_\beta z(\alpha)|^2} - \frac{1}{|z_\alpha(\alpha)|^2\beta^2} + \frac{z_\alpha\cdot z_{\alpha\alpha}}{|z_\alpha|^4\beta}\right)\right| \leq \|z\|^k_{C^2}\frac{1}{\beta^{1/2}}\|z\|_{C^{2+\frac{1}{2}}}\|F(z)\|^k_{L^\infty}$$

and that

$$\left|\frac{1}{|\Delta_\beta z(\alpha)|^2} - \frac{1}{|z_\alpha(\alpha)|^2\beta^2} + \frac{z_\alpha\cdot z_{\alpha\alpha}}{|z_\alpha|^4\beta} - \left(\frac{1}{|\Delta_\beta x(\alpha)|^2} - \frac{1}{|x_\alpha(\alpha)|^2\beta^2} + \frac{x_\alpha\cdot x_{\alpha\alpha}}{|x_\alpha|^4\beta}\right)\right|$$
$$\leq \|z\|^k_{C^2}\|x\|^k_{C^2}\frac{1}{\beta^{1/2}}\|D\|_{C^{2+\frac{1}{2}}}\|F(z)\|^k_{L^\infty}\|F(x)\|^k_{L^\infty}$$

to find

$$I^2_{1,2,2} \leq \frac{1}{8\pi^2}\int_{-\pi}^{\pi}\frac{Q_z^2}{|z_\alpha|^2}\sigma_z|\partial_\alpha^4 D|^2$$
$$+ C\|Q_z\|^6_{L^\infty}\|\sigma_z\|_{L^\infty}\|z\|^k_{C^2}\|x\|^k_{C^2}\|D\|_{C^{2+\frac{1}{2}}}\|\partial_\alpha^4 x\|^2_{L^2}\|F(z)\|^k_{L^\infty}\|F(x)\|^k_{L^\infty}$$

We've used that

$$\left(\int_{-\pi}^{\pi}d\alpha\left(\int_{-\pi}^{\pi}\frac{\partial_\alpha^4 x(\alpha-\beta)}{|\beta|^{1/2}}d\beta\right)^2\right)^{1/2} \leq C\|\partial_\alpha^4 x\|_{L^2}.$$

We split further in $I^1_{1,2,2} = I^{1,1}_{1,2,2} + I^{1,2}_{1,2,2}$:

$$\begin{aligned}
I^{1,1}_{1,2,2} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{Q_z^4}{|z_\alpha|^2}\sigma_z\partial_\alpha^4 D\int_{-\pi}^{\pi}\Delta_\beta\partial_\alpha^4 x^\perp(\alpha)\left(\frac{1}{|\Delta_\beta z(\alpha)|^2}-\frac{1}{|\Delta_\beta x(\alpha)|^2}\right)\\
&\quad\times(\omega(\alpha-\beta)-\omega(\alpha)+\omega_\alpha(\alpha)\beta)\,d\beta d\alpha\\
I^{1,2}_{1,2,2} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{Q_z^4}{|z_\alpha|^2}\sigma_z\partial_\alpha^4 D\omega_\alpha(\alpha)\int_{-\pi}^{\pi}\Delta_\beta\partial_\alpha^4 x^\perp(\alpha)\left(\frac{\beta}{|\Delta_\beta z(\alpha)|^2}-\frac{\beta}{|\Delta_\beta x(\alpha)|^2}\right)d\beta d\alpha
\end{aligned}$$

Inside of the β integral in $I^{1,1}_{1,2,2}$ there is no principal value, so the appropriate estimate follows:

$$I^{1,1}_{1,2,2} \leq CP(E(t))$$

For $I^{1,2}_{1,2,2}$ se proceed as for $I^2_{1,2,2}$, se decompose adding and subtracting $\dfrac{1}{|z_\alpha|^2\beta}-\dfrac{1}{|x_\alpha|^2\beta}$. Thus, we are done with $I_{1,2,2}$. We decompose $I_{1,2,3} = I^1_{1,2,3} + I^2_{1,2,3} + I^3_{1,2,3}$.

$$\begin{aligned}
I^1_{1,2,3} &= \int_{-\pi}^{\pi}\frac{Q_z^2}{|z_\alpha|^2}\sigma_z\partial_\alpha^4 DQ_z^2\frac{1}{2\pi}\int_{-\pi}^{\pi}\Delta_\beta\partial_\alpha^4 x^\perp(\alpha)\\
&\quad\times\left(\frac{1}{|\Delta_\beta x(\alpha)|^2}-\frac{1}{|x_\alpha|^2\beta^2}+\frac{x_\alpha\cdot x_{\alpha\alpha}}{|x_\alpha|^4\beta}\right)d(\alpha-\beta)d\beta d\alpha\\
I^2_{1,2,3} &= -\int_{-\pi}^{\pi}\frac{Q_z^2}{|z_\alpha|^2}\sigma_z\partial_\alpha^4 DQ_z^2\frac{\partial_\alpha^4 x^\perp(\alpha)}{|x_\alpha|^2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\Delta_\beta d(\alpha)}{\beta^2}d\beta d\alpha\\
I^3_{1,2,3} &= \int_{-\pi}^{\pi}\frac{Q_z^2}{|z_\alpha|^2}\sigma_z\partial_\alpha^4 DQ_z^2\frac{1}{|x_\alpha|^2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\Delta_\beta(d\partial_\alpha^4 x^\perp)(\alpha)}{\beta^2}d\beta d\alpha
\end{aligned}$$

It's easy to obtain:

$$\begin{aligned}
I^1_{1,2,3} &\leq \frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{Q_z^2}{|z_\alpha|^2}\sigma_z|\partial_\alpha^4 D|^2 d\alpha + C\|Q_z\|^6_{L^\infty}\|\sigma_z\|_{L^\infty}\|d\|_{L^\infty}\|x\|^k_{C^2}\|F(x)\|^k_{L^\infty}\|x\|_{C^{2,\delta}}\|\partial_\alpha^4 x\|^2_{L^2}\\
&\leq CP(E(t))\\
I^2_{1,2,3} &\leq CP(E(t)) \text{ analogously since } \|\Lambda d\|_{L^\infty} \leq C\|d\|_{H^2}\\
I^3_{1,2,3} &\leq CP(E(t)) \text{ using } \|\Lambda(d\partial_\alpha^4 x^\perp)\|_{L^2} \leq C\|d\|_{H^2}\|x\|_{H^5}.
\end{aligned}$$

We are done with $I_{1,2,3}$. To deal with $I_{1,2,4}$ se use that

$$Q_z^2 - Q_x^2 = 2Q((1-t)z+tx)\nabla Q((1-t)z+tx)\cdot D(\alpha) \text{ for } t\in(0,1).$$

Then it is easy to find

$$I_{1,2,4} \leq CP(E(t)),$$

and we are done with $I_{1,2}$. We decompose $I_{1,3}$ as

$$
\begin{aligned}
I_{1,3} &= I_{1,3,1} + I_{1,3,2} + I_{1,3,3} + I_{1,3,4} + I_{1,3,5} + I_{1,3,6} \\
I_{1,3,1} &= \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2}\sigma_z \partial_\alpha^4 D Q_z^2 \frac{-1}{\pi}\int_{-\pi}^{\pi} \frac{\Delta_\beta z^\perp(\alpha)}{|\Delta_\beta z(\alpha)|^4}\Delta_\beta z(\alpha)\cdot\Delta_\beta\partial_\alpha^4 D(\alpha)\omega(\alpha-\beta)d\beta d\alpha \\
I_{1,3,2} &= \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2}\sigma_z \partial_\alpha^4 D Q_z^2 \frac{-1}{\pi}\int_{-\pi}^{\pi} \frac{\Delta_\beta z^\perp(\alpha)}{|\Delta_\beta z(\alpha)|^4}\Delta_\beta z(\alpha)\cdot\Delta_\beta\partial_\alpha^4 x(\alpha)d(\alpha-\beta)d\beta d\alpha \\
I_{1,3,3} &= \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2}\sigma_z \partial_\alpha^4 D Q_z^2 \frac{-1}{\pi}\int_{-\pi}^{\pi} \frac{\Delta_\beta z^\perp(\alpha)}{|\Delta_\beta z(\alpha)|^4}\Delta_\beta D\cdot\Delta_\beta\partial_\alpha^4 x(\alpha)\gamma(\alpha-\beta)d\beta d\alpha \\
I_{1,3,4} &= \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2}\sigma_z \partial_\alpha^4 D Q_z^2 \frac{-1}{\pi}\int_{-\pi}^{\pi} \frac{\Delta_\beta D^\perp(\alpha)}{|\Delta_\beta z(\alpha)|^4}\Delta_\beta x(\alpha)\cdot\Delta_\beta\partial_\alpha^4 x(\alpha)\gamma(\alpha-\beta)d\beta d\alpha \\
I_{1,3,5} &= \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2}\sigma_z \partial_\alpha^4 D Q_z^2 \frac{-1}{\pi}\int_{-\pi}^{\pi} \Delta_\beta x^\perp(\alpha)\Delta_\beta x(\alpha)\cdot\Delta_\beta\partial_\alpha^4 x(\alpha)\gamma(\alpha-\beta) \\
&\quad \times\left(\frac{1}{|\Delta_\beta z(\alpha)|^4}-\frac{1}{|\Delta_\beta x(\alpha)|^4}\right)d\beta d\alpha \\
I_{1,3,6} &= \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2}\sigma_z \partial_\alpha^4 D (Q_z^2-Q_x^2) \frac{-1}{\pi}\int_{-\pi}^{\pi} \frac{\Delta_\beta x^\perp(\alpha)}{|\Delta_\beta x(\alpha)|^4}\Delta_\beta x(\alpha)\cdot\Delta_\beta\partial_\alpha^4 x(\alpha)\gamma(\alpha-\beta)d\beta d\alpha
\end{aligned}
$$

$I_{1,3,j}$, $\quad j=2,3,4,5,6$ are easier to deal with (It can be done as before). Therefore we focus on $I_{1,3,1}$.

$$
\begin{aligned}
I_{1,3,1} &= I_{1,3,1}^1 + I_{1,3,1}^2 + I_{1,3,1}^3 \\
I_{1,3,1}^1 &= \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2}\sigma_z \partial_\alpha^4 D Q_z^2 \frac{-1}{\pi}\int_{-\pi}^{\pi}\left(\frac{\Delta_\beta z^\perp(\alpha)}{|\Delta_\beta z(\alpha)|^4}\Delta_\beta z(\alpha)\omega(\alpha-\beta)\right. \\
&\quad \left.-\frac{\partial_\alpha z^\perp(\alpha)}{|\partial_\alpha z(\alpha)|^4}\partial_\alpha z(\alpha-\beta)\omega(\alpha)\frac{1}{\beta^2}\right)\cdot\Delta_\beta\partial_\alpha^4 D(\alpha)d\beta d\alpha \\
I_{1,3,1}^2 &= \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2}\sigma_z \partial_\alpha^4 D Q_z^2 \frac{-1}{\pi}\frac{\partial_\alpha z^\perp(\alpha)}{|\partial_\alpha z(\alpha)|^4}\omega(\alpha)\partial_\alpha^4 D(\alpha)\cdot\int_{-\pi}^{\pi}\frac{\partial_\alpha z(\alpha-\beta)-\partial_\alpha z(\alpha)}{\beta^2}d\beta d\alpha \\
I_{1,3,1}^3 &= \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2}\sigma_z \partial_\alpha^4 D Q_z^2 \frac{-1}{\pi}\frac{\partial_\alpha z^\perp(\alpha)}{|\partial_\alpha z(\alpha)|^4}\omega(\alpha)\int_{-\pi}^{\pi}\frac{\Delta_\beta(\partial_\alpha z\cdot\partial_\alpha^4 D)(\alpha)}{\beta^2}d\beta d\alpha
\end{aligned}
$$

In $I_{1,3,1}^1$ we find a commutator, which can be handled as before. It is also easy to estimate $I_{1,3,1}^2$.

To deal with $I_{1,3,1}^3$ we remember that

$$
\begin{aligned}
\partial_\alpha z(\alpha)\cdot\partial_\alpha^4 D(\alpha) &= \partial_\alpha z(\alpha)\cdot\partial_\alpha^4 z(\alpha) - \partial_\alpha x(\alpha)\cdot\partial_\alpha^4 x(\alpha) - \partial_\alpha D(\alpha)\cdot\partial_\alpha^4 x(\alpha) \\
&= -3\partial_\alpha^2 z(\alpha)\cdot\partial_\alpha^3 z(\alpha) + 3\partial_\alpha^2 x(\alpha)\partial_\alpha^3 x(\alpha) - \partial_\alpha D(\alpha)\cdot\partial_\alpha^4 x(\alpha)
\end{aligned}
$$

That allows us to decompose further

$$\partial_\alpha z(\alpha)\cdot\partial_\alpha^4 D(\alpha) = -3\partial_\alpha^2 z(\alpha)\cdot\partial_\alpha^3 D(\alpha) - 3\partial_\alpha^2 D(\alpha)\partial_\alpha^3 x(\alpha) - \partial_\alpha D(\alpha)\cdot\partial_\alpha^4 x(\alpha)$$

which yields

$$\begin{aligned}
I_{1,3,1}^3 &= I_{1,3,1}^{3,1} + I_{1,3,1}^{3,2} + I_{1,3,1}^{3,3}\\
I_{1,3,1}^{3,1} &= \frac{3}{\pi}\int_{-\pi}^{\pi}\frac{Q_z^4}{|z_\alpha|^2}\sigma_z\partial_\alpha^4 D\cdot\frac{\partial_\alpha z^\perp(\alpha)}{|\partial_\alpha z(\alpha)|^4}\omega(\alpha)\int_{-\pi}^{\pi}\frac{\Delta_\beta(\partial_\alpha^2 z\cdot\partial_\alpha^3 D)(\alpha)}{\beta^2}d\beta d\alpha\\
I_{1,3,1}^{3,2} &= \frac{3}{\pi}\int_{-\pi}^{\pi}\frac{Q_z^4}{|z_\alpha|^2}\sigma_z\partial_\alpha^4 D\cdot\frac{\partial_\alpha z^\perp(\alpha)}{|\partial_\alpha z(\alpha)|^4}\omega(\alpha)\int_{-\pi}^{\pi}\frac{\Delta_\beta(\partial_\alpha^2 D\cdot\partial_\alpha^3 x)(\alpha)}{\beta^2}d\beta d\alpha\\
I_{1,3,1}^{3,3} &= \frac{3}{\pi}\int_{-\pi}^{\pi}\frac{Q_z^4}{|z_\alpha|^2}\sigma_z\partial_\alpha^4 D\cdot\frac{\partial_\alpha z^\perp(\alpha)}{|\partial_\alpha z(\alpha)|^4}\omega(\alpha)\int_{-\pi}^{\pi}\frac{\Delta_\beta(\partial_\alpha D\cdot\partial_\alpha^4 x)(\alpha)}{\beta^2}d\beta d\alpha
\end{aligned}$$

We use that

$$\left\|\int_{-\pi}^{\pi}\frac{\Delta_\beta(\partial_\alpha^2 z\cdot\partial_\alpha^3 D)(\alpha)}{\beta^2}d\beta\right\|_{L^2}^2 \leq C\left\|\partial_\alpha(\partial_\alpha^2 z\cdot\partial_\alpha^3 D)\right\|_{L^2}^2 \leq CP(E(t))$$

to control $I_{1,3,1}^{3,1}$. $I_{1,3,1}^{3,2}$ follows similarly. We control $I_{1,3,1}^{3,3}$ using that

$$\begin{aligned}
\left\|\int_{-\pi}^{\pi}\frac{\Delta_\beta(\partial_\alpha D\cdot\partial_\alpha^4 x)(\alpha)}{\beta^2}d\beta\right\|_{L^2}^2 &\leq \left\|\partial_\alpha(\partial_\alpha D\cdot\partial_\alpha^4 x)\right\|_{L^2}^2\\
&\leq \|\partial_\alpha D\|_{L^\infty}^2\|\partial_\alpha^5 x\|_{L^2}^2 + \|\partial_\alpha^2 D\|_{L^\infty}^2\|\partial_\alpha^4 x\|_{L^2}^2 \leq CP(E(t))
\end{aligned}$$

This allows us to finish the estimates for $I_{1,3,1}^{3,3}$ and $I_{1,3,1}^3$. We are done with $I_{1,3,1}$ and $I_{1,3}$. We now decompose $I_{1,4}$.

$$\begin{aligned}
I_{1,4} &= I_{1,4,1} + I_{1,4,2} + I_{1,4,3} + I_{1,4,4}\\
I_{1,4,1} &= \int_{-\pi}^{\pi}\frac{Q_z^2}{|z_\alpha|^2}\sigma_z\partial_\alpha^4 D\cdot Q_z^2 BR(z,\partial_\alpha^4 d)d\beta d\alpha\\
I_{1,4,2} &= \int_{-\pi}^{\pi}\frac{Q_z^2}{|z_\alpha|^2}\sigma_z\partial_\alpha^4 D\cdot Q_z^2\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\Delta_\beta D^\perp(\alpha)}{|\Delta_\beta z(\alpha)|^2}\partial_\alpha^4\gamma(\alpha-\beta)d\beta d\alpha\\
I_{1,4,3} &= \int_{-\pi}^{\pi}\frac{Q_z^2}{|z_\alpha|^2}\sigma_z\partial_\alpha^4 D\cdot Q_z^2\frac{1}{2\pi}\int_{-\pi}^{\pi}\Delta_\beta x^\perp(\alpha)\partial_\alpha^4\gamma(\alpha-\beta)\left(\frac{1}{|\Delta_\beta z(\alpha)|^2}-\frac{1}{|\Delta_\beta x(\alpha)|^2}\right)d\beta d\alpha\\
I_{1,4,4} &= \int_{-\pi}^{\pi}\frac{Q_z^2}{|z_\alpha|^2}\sigma_z\partial_\alpha^4 D\cdot(Q_z^2-Q_x^2)BR(x,\partial_\alpha^4\gamma)d\alpha
\end{aligned}$$

We control $I_{1,4,2}$, $I_{1,4,3}$ and $I_{1,4,4}$ as before. We further split

$$
\begin{aligned}
I_{1,4,1} &= I^1_{1,4,1} + I^2_{1,4,1} + I_{1,4,3} + I_{1,4,4}\\
I^1_{1,4,1} &= \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2}\sigma_z \partial_\alpha^4 D \cdot Q_z^2 \left(BR(z, \partial_\alpha^4 d) - \frac{1}{2}\frac{\partial_\alpha z^\perp(\alpha)}{|\partial_\alpha z(\alpha)|^2} H(\partial_\alpha^4 d)\right) d\alpha\\
I^2_{1,4,1} &= \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2}\sigma_z \partial_\alpha^4 D \cdot \frac{\partial_\alpha z^\perp(\alpha)}{|z_\alpha|} \left(\frac{Q_z^2}{2|z_\alpha|} H(\partial_\alpha^4 d) - H\left(\frac{Q_z^2}{2|z_\alpha|}\partial_\alpha^4 d\right)\right) d\alpha\\
I^3_{1,4,1} &= -\int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2}\sigma_z \partial_\alpha^4 D \cdot \frac{\partial_\alpha z^\perp(\alpha)}{|z_\alpha|} H\left(\left(\frac{Q_z^2}{2|z_\alpha|} - \frac{Q_x^2}{2|x_\alpha|}\right)\partial_\alpha^4 \gamma\right) d\alpha\\
I^4_{1,4,1} &= \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2}\sigma_z \partial_\alpha^4 D \cdot \frac{\partial_\alpha z^\perp(\alpha)}{|z_\alpha|} H\left(\frac{Q_z^2 \partial_\alpha^4 \omega}{2|z_\alpha|} - \frac{Q_x^2 \partial_\alpha^4 \gamma}{2|x_\alpha|}\right) d\alpha
\end{aligned}
$$

There are commutators in $I^1_{1,4,1}$ and $I^2_{1,4,1}$ so they are easy to estimate. To get the estimate for $I^3_{1,4,1}$ we bound

$$
\left\| H\left(\left(\frac{Q_z^2}{2|z_\alpha|} - \frac{Q_x^2}{2|x_\alpha|}\right)\partial_\alpha^4 \gamma\right)\right\|_{L^2}^2 \leq \underbrace{\left\| \frac{Q_z^2}{2|z_\alpha|} - \frac{Q_x^2}{2|x_\alpha|}\right\|_{L^\infty}^2}_{\text{at the level of } D(\alpha)} \|\partial_\alpha^4 \gamma\|_{L^2}^2 \leq CE^2(t)
$$

We now remember the following formulas:

$$
\begin{aligned}
\varphi &= \frac{Q_z^2 \omega}{2|z_\alpha|} - c|z_\alpha|\\
\psi &= \frac{Q_x^2 \gamma}{2|x_\alpha|} - b_s|x_\alpha|
\end{aligned}
$$

These yield

$$
I^4_{1,4,1} = S + I^{4,1}_{1,4,1} + I^{4,2}_{1,4,1} + \text{ l.o.t},
$$

where

$$
\begin{aligned}
S &= \int_{-\pi}^{\pi} Q_z^2 \sigma_z \partial_\alpha^4 D \cdot \frac{\partial_\alpha z^\perp(\alpha)}{|z_\alpha|^3} H(\partial_\alpha^4 \mathcal{D})(\alpha) d\alpha\\
I^{4,1}_{1,4,1} &= -\int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2}\sigma_z \partial_\alpha^4 D \cdot \frac{\partial_\alpha z^\perp(\alpha)}{|z_\alpha|} H\left(\frac{Q_z \nabla Q(z)\cdot \partial_\alpha^4 z}{|z_\alpha|}\omega - \frac{Q_x \nabla Q(x)\cdot \partial_\alpha^4 x}{|x_\alpha|}\gamma\right) d\alpha\\
I^{4,2}_{1,4,1} &= \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2}\sigma_z \partial_\alpha^4 D \cdot \frac{\partial_\alpha z^\perp(\alpha)}{|z_\alpha|} H\left(\partial_\alpha^4 \left(c|z_\alpha| - b_s|x_\alpha|\right)\right) d\alpha
\end{aligned}
$$

S is going to appear later with a negative sign and therefore cancel out. $I^{4,1}_{1,4,1}$ can be bounded as before since it is low order.

We show how to deal with $I^{4,2}_{1,4,1}$. We compute

$$\partial^4_\alpha(c|z_\alpha|) = -\partial^3_\alpha\left((Q^2_z BR)_\alpha \cdot \frac{z_\alpha}{|z_\alpha|}\right); \quad \partial^4_\alpha(b_s|x_\alpha|) = -\partial^3_\alpha\left((Q^2_x BR)_\alpha \frac{x_\alpha}{|x_\alpha|}\right)$$

Then, in $\partial^4_\alpha(c|z_\alpha|) - \partial^4_\alpha(b_s|x_\alpha|)$ we consider the most singular terms

$$\begin{aligned}
\partial^4_\alpha(c|z_\alpha|) - \partial^4_\alpha(b_s|x_\alpha|) &= J_1 + J_2 + J_3 + J_4 + J_5 + \text{ l.o.t.} \\
J_1 &= -2Q_z\nabla Q(z)\cdot\partial^4_\alpha z BR(z,\omega)\cdot\frac{z_\alpha}{|z_\alpha|} + 2Q_x\nabla Q(x)\cdot\partial^4_\alpha x BR(x,\gamma)\cdot\frac{x_\alpha}{|x_\alpha|} \\
J_2 &= -(Q^2_z BR)_\alpha\frac{\partial^4_\alpha z}{|z_\alpha|} + (Q^2_x BR)_\alpha\frac{\partial^4_\alpha x}{|x_\alpha|} \\
J_3 &= Q^2_z\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\Delta_\beta z^\perp(\alpha)}{|\Delta_\beta z(\alpha)|^4}\frac{z_\alpha(\alpha)}{|z_\alpha|}\Delta_\beta z(\alpha)\cdot\Delta_\beta\partial^4_\alpha z(\alpha)\omega(\alpha-\beta)d\beta \\
&\quad - Q^2_x\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\Delta_\beta x^\perp(\alpha)}{|\Delta_\beta x(\alpha)|^4}\frac{x_\alpha(\alpha)}{|x_\alpha|}\Delta_\beta x(\alpha)\cdot\Delta_\beta\partial^4_\alpha x(\alpha)\gamma(\alpha-\beta)d\beta \\
J_4 &= -Q^2_z BR(z,\partial^4_\alpha\omega)\cdot\frac{z_\alpha}{|z_\alpha|} + Q^2_x BR(x,\partial^4_\alpha\gamma)\cdot\frac{x_\alpha}{|x_\alpha|}
\end{aligned}$$

J_5 will be given later. In J_1 and J_2 we find 4th order terms in derivatives in z and x so they are fine. In J_3 we find inside the integrals

$$\Delta_\beta z^\perp(\alpha)\cdot z_\alpha(\alpha) = (z(\alpha)-z(\alpha-\beta)-\beta z_\alpha(\alpha))^\perp\cdot z_\alpha(\alpha) \tag{5.9}$$

$$\Delta_\beta x^\perp(\alpha)\cdot x_\alpha(\alpha) = (x(\alpha)-x(\alpha-\beta)-\beta x_\alpha(\alpha))^\perp\cdot x_\alpha(\alpha) \tag{5.10}$$

This implies that we find "Hilbert" transforms applied to four derivatives of x and z. We are done with J_3.

In J_4 we also find them inside the integrals (5.9) and (5.10) so it is easy to check that we have kernels of degree 0 applied to four derivatives of $\partial^4_\alpha\omega$ and $\partial^4_\alpha\gamma$. This implies that we have a Hilbert transform applied to $\partial^3_\alpha\omega$ and $\partial^3_\alpha\gamma$ so we are done with J_4. The most dangerous term is J_5 which is given by

$$J_5 = -Q^2_z\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\Delta_\beta\partial^4_\alpha z^\perp(\alpha)}{|\Delta_\beta z(\alpha)|^2}\cdot\frac{z_\alpha(\alpha)}{|z_\alpha|}\omega(\alpha-\beta)d\beta + Q^2_x\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\Delta_\beta\partial^4_\alpha x^\perp(\alpha)}{|\Delta_\beta x(\alpha)|^2}\cdot\frac{x_\alpha(\alpha)}{|x_\alpha|}\gamma(\alpha-\beta)d\beta$$

We split further

$$\begin{aligned}
J_5 &= J_{5,1} + J_{5,2} \\
J_{5,1} &= -Q^2_z\frac{1}{2\pi}\frac{z_\alpha(\alpha)}{|z_\alpha|}\cdot\int_{-\pi}^{\pi}\left(\frac{\omega(\alpha-\beta)}{|\Delta_\beta z(\alpha)|^2} - \frac{\omega(\alpha)}{|z_\alpha|^2 4\sin^2(\beta/2)}\right)\Delta_\beta\partial^4_\alpha z^\perp(\alpha)d\beta \\
&\quad + Q^2_x\frac{1}{2\pi}\frac{x_\alpha(\alpha)}{|x_\alpha|}\cdot\int_{-\pi}^{\pi}\left(\frac{\gamma(\alpha-\beta)}{|\Delta_\beta x(\alpha)|^2} - \frac{\gamma(\alpha)}{|x_\alpha|^2 4\sin^2(\beta/2)}\right)\Delta_\beta\partial^4_\alpha x^\perp(\alpha)d\beta \\
J_{5,2} &= -Q^2_z\frac{1}{2}\frac{z_\alpha(\alpha)}{|z_\alpha|^3}\omega(\alpha)\cdot\Lambda(\partial^4_\alpha z^\perp)(\alpha) + Q^2_x\frac{1}{2}\frac{x_\alpha(\alpha)}{|x_\alpha|^3}\gamma(\alpha)\cdot\Lambda(\partial^4_\alpha x^\perp)(\alpha)
\end{aligned}$$

In $J_{5,1}$ we find a Hilbert transform applied to $\partial_\alpha^4 z^\perp$ and $\partial_\alpha^4 x^\perp$ so it is fine. We split further:

$$\begin{aligned}
J_{5,2} &= J_{5,2,1} + J_{5,2,2} + J_{5,2,3}\\
J_{5,2,1} &= \left(Q_x^2 \frac{1}{2}\frac{x_\alpha(\alpha)}{|x_\alpha|^3}\gamma(\alpha) - Q_z^2 \frac{1}{2}\frac{z_\alpha(\alpha)}{|z_\alpha|^3}\omega(\alpha)\right)\cdot \Lambda(\partial_\alpha^4 x^\perp)(\alpha)\\
J_{5,2,2} &= \Lambda\left(Q_z^2\frac{1}{2}\frac{z_\alpha(\alpha)}{|z_\alpha|^3}\omega(\alpha)\partial_\alpha^4 D^\perp\right) - Q_z^2\frac{1}{2}\frac{z_\alpha(\alpha)}{|z_\alpha|^3}\omega(\alpha)\Lambda(\partial_\alpha^4 D^\perp)\\
J_{5,2,3} &= -\Lambda\left(Q_z^2\frac{1}{2}\frac{z_\alpha(\alpha)}{|z_\alpha|^3}\omega(\alpha)\partial_\alpha^4 D^\perp\right)
\end{aligned}$$

$J_{5,2,1}$ can be estimated as before (there are more derivatives: 5 in total, but they are in x). In $J_{5,2,2}$ we find a commutator. Finally:

$$I_{1,4,1}^{4,2} \leq CP(E(t)) - \int_{-\pi}^{\pi} Q_z^2 \sigma_z \partial_\alpha^4 D \cdot \frac{z^\perp(\alpha)}{|z_\alpha|} H\left(\Lambda\left(Q_z^2\frac{1}{2}\frac{z_\alpha}{|z_\alpha|^3}\omega\partial_\alpha^4 D^\perp\right)\right) d\alpha$$

We use that $H(\Lambda) = -\partial_\alpha$ and $z_\alpha \cdot \partial_\alpha^4 D^\perp = -z_\alpha^\perp \cdot \partial_\alpha^4 D$ to obtain:

$$\begin{aligned}
I_{1,4,1}^{4,2} &\leq CP(E(t)) - \frac{1}{2}\int_{-\pi}^{\pi} Q_z^2\sigma_z\partial_\alpha^4 D\cdot\frac{z^\perp(\alpha)}{|z_\alpha|}\partial_\alpha\left(\frac{Q_z^2\omega}{|z_\alpha|^2}\partial_\alpha^4 D\cdot\frac{z_\alpha^\perp}{|z_\alpha|}\right)d\alpha\\
&\leq CP(E(t)) \underbrace{- \frac{1}{2}\int_{-\pi}^{\pi} Q_z^2\sigma_z\partial_\alpha^4 D\cdot\frac{z^\perp(\alpha)}{|z_\alpha|}\partial_\alpha^4 D\cdot\frac{z^\perp(\alpha)}{|z_\alpha|}\partial_\alpha\left(\frac{Q_z^2\omega}{|z_\alpha|}\right)d\alpha}_{\text{Easy to estimate by } CP(E(t))}\\
&\quad - \frac{1}{2}\int_{-\pi}^{\pi} Q_z^2\sigma_z\frac{Q_z^2\omega}{|z_\alpha|^2}\underbrace{\partial_\alpha^4 D\cdot\frac{z^\perp(\alpha)}{|z_\alpha|}\partial_\alpha\left(\partial_\alpha^4 D\cdot\frac{z_\alpha^\perp}{|z_\alpha|}\right)}_{\text{Integration by parts}}d\alpha
\end{aligned}$$

Then we are done with $I_{1,4,1}^{4,2}$, $I_{1,4,1}^{4}$, $I_{1,4,1}$, $I_{1,4}$ and I_1.
To finish with I it remains to control I_2. We split it as:

$$\begin{aligned}
I_2 &= I_{2,1} + I_{2,2} + \text{ l.o.t}\\
I_{2,1} &= \int_{-\pi}^{\pi}\frac{Q_z^2}{|z_\alpha|^2}\sigma_z\partial_\alpha^4 D(c\partial_\alpha^5 z - b\partial_\alpha^5 x)d\alpha\\
I_{2,2} &= \int_{-\pi}^{\pi}\frac{Q_z^2}{|z_\alpha|^2}\sigma_z\partial_\alpha^4 D(\partial_\alpha^4 c z_\alpha - \partial_\alpha^4 b x_\alpha)d\alpha
\end{aligned}$$

The low order terms are easier to deal with. We further split $I_{2,1}$.

$$
\begin{aligned}
I_{2,1} &= I_{2,1,1} + I_{2,1,2} + I_{2,1,3} \\
I_{2,1,1} &= \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2}\sigma_z c \underbrace{\partial_\alpha^4 D \partial_\alpha^5 D}_{\text{Integration by parts}} d\alpha \\
I_{2,1,2} &= \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2}\sigma_z \partial_\alpha^4 D(c-b_s) \underbrace{\partial_\alpha^5 x}_{\text{5 derivatives, but in } x} d\alpha \\
I_{2,1,3} &= \underbrace{\int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2}\sigma_z \partial_\alpha^4 D b_e \partial_\alpha^5 x \, d\alpha}_{\text{Error term}}
\end{aligned}
$$

We find $I_{2,1} \leq CP(E(t)) + c\delta(t)$. We decompose $I_{2,2}$.

$$
\begin{aligned}
I_{2,2} &= I_{2,2,1} + I_{2,2,2} + I_{2,2,3} \\
I_{2,2,1} &= \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2}\sigma_z \partial_\alpha^4 D(\partial_\alpha^4 c - \partial_\alpha^4 b_s)\cdot z_\alpha d\alpha \\
I_{2,2,2} &= \int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2}\sigma_z \partial_\alpha^4 D \underbrace{\partial_\alpha^4 b_s}_{\text{5 derivatives in } x} \partial_\alpha D d\alpha \\
I_{2,2,3} &= \underbrace{-\int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2}\sigma_z \partial_\alpha^4 D \partial_\alpha^4 b_e x_\alpha d\alpha}_{\text{Error term}}
\end{aligned}
$$

We deal with $I_{2,2,1}$ more carefully. We use that

$$
\begin{aligned}
\partial_\alpha^4 D \cdot z_\alpha &= \partial_\alpha^4 z \cdot z_\alpha - \partial_\alpha^4 x \cdot x_\alpha - \partial_\alpha^4 x \cdot D_\alpha \\
&= -3\partial_\alpha^3 z \cdot \partial_\alpha^2 z + 3\partial_\alpha^3 x \cdot \partial_\alpha^2 x - \partial_\alpha^4 x \cdot D_\alpha \\
&= -3\partial_\alpha^3 D \cdot \partial_\alpha^2 z - 3\partial_\alpha^3 x \cdot \partial_\alpha^2 D - \partial_\alpha^4 x \cdot D_\alpha
\end{aligned}
$$

to obtain

$$
\begin{aligned}
I_{2,2,1} &= I_{2,2,1}^1 + I_{2,2,1}^2 + I_{2,2,1}^3 \\
I_{2,2,1}^1 &= -3\int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2}\sigma_z \partial_\alpha^3 D \cdot \partial_\alpha^2 z \partial_\alpha^4 (c-b_s) d\alpha \\
I_{2,2,1}^2 &= -3\int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2}\sigma_z \partial_\alpha^3 x \cdot \partial_\alpha^2 D \partial_\alpha^4 (c-b_s) d\alpha \\
I_{2,2,1}^3 &= -\int_{-\pi}^{\pi} \frac{Q_z^2}{|z_\alpha|^2}\sigma_z \partial_\alpha^4 x \cdot \partial_\alpha D \partial_\alpha^4 (c-b_s) d\alpha
\end{aligned}
$$

We can integrate by parts in all of the above terms to get low order terms. We are finally done with I.

5.4.2 Computing the difference $\varphi - \psi$

From the local existence proof we find the equation for φ_t:

$$\left\{\begin{array}{rl} \varphi_t & = -\varphi B_z(t) - \dfrac{Q_z^2}{2|z_{al}|}\partial_\alpha\left(\dfrac{\varphi^2}{Q_z^2}\right) - Q_z^2\left(BR_t\cdot\dfrac{z_\alpha}{|z_\alpha|} + \dfrac{(P_2^{-1}(z))_\alpha}{|z_\alpha|}\right) \\ & +Q_zQ_t^z\dfrac{\omega}{|z_\alpha|} - 2cBR\cdot\dfrac{z_\alpha}{|z_\alpha|}Q_zQ_\alpha^z - c^2|z_\alpha|\dfrac{Q_\alpha^z}{Q_z} - \dfrac{Q_z^3}{|z_\alpha|}|BR|^2Q_\alpha^z - (c|z_\alpha|)_t \\ B_z(t) & = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}(Q_z^2BR)_\alpha\cdot\dfrac{z_\alpha}{|z_\alpha|}d\alpha \end{array}\right. \tag{5.11}$$

We will show how to find the equation for ψ_t. We start from

$$\psi = \frac{Q_x^2\gamma}{2|x_\alpha|} - b_s|x_\alpha|$$

and therefore

$$\frac{\psi^2}{Q_x^2} = \frac{Q_x^2\gamma^2}{4|x_\alpha|} + \frac{b_s^2|x_\alpha|^2}{Q_x^2} - \gamma b_s,$$

that yields

$$-\partial_\alpha\left(\frac{\psi^2}{Q_x^2}\right) = -\partial_\alpha\left(\frac{Q_x^2\gamma^2}{4|x_\alpha|}\right) - \partial_\alpha\left(\frac{b_s^2|x_\alpha|^2}{Q_x^2}\right) + \partial_\alpha\left(\gamma b_s\right)$$

The equation for γ_t reads:

$$\begin{aligned}\gamma_t =& -2BR_t\cdot x_\alpha - (Q_x^2)_\alpha|BR|^2 + 2b_sBR_\alpha\cdot x_\alpha \\ & -\partial_\alpha\left(\frac{\psi^2}{Q_x^2}\right) + \left(\frac{b_s^2|x_\alpha|^2}{Q_x^2}\right)_\alpha - 2(P_2^{-1}(z))_\alpha + 2b_eBR_\alpha\cdot x_\alpha + (b_e\gamma)_\alpha + g\end{aligned}$$

Then

$$\begin{aligned}\omega_t =& Q_x(Q_x)_t\frac{\gamma}{|x_\alpha}| - \frac{Q_x^2\gamma}{2|x_\alpha|^3}x_\alpha\cdot x_{\alpha t} + \frac{Q_x^2\gamma_t}{2|x_\alpha|} - (b_s|x_\alpha|)_t \\ =& Q_x(Q_x)_t\frac{\gamma}{|x_\alpha|} - \frac{Q_x^2\gamma}{2|x_\alpha|}B_x(t) - \frac{Q_x^2\gamma}{2|x_\alpha|}\frac{1}{2\pi}\int_{-\pi}^{\pi}f_\alpha\cdot\frac{x_\alpha}{|x_\alpha|^2}d\alpha \\ & + \frac{Q_x^2}{2|x_\alpha|}\Bigg(-2BR_t\cdot x_\alpha - (Q_x^2)_\alpha|BR|^2 + 2b_sBR_\alpha\cdot x_\alpha - \partial_\alpha\left(\frac{\psi^2}{Q_x^2}\right) + \left(\frac{b_s^2|x_\alpha|^2}{Q_x^2}\right)_\alpha \\ & -2(P_2^{-1}(z))_\alpha + 2b_eBR_\alpha\cdot x_\alpha + (b_e\gamma)_\alpha + g\Bigg) - (b_s|x_\alpha|)_t\end{aligned}$$

We should remark that we have used that

$$x_\alpha\cdot x_{\alpha t} = \frac{1}{2\pi}\int_{-\pi}^{\pi}(Q_x^2BR)_\alpha\cdot x_\alpha d\alpha + \frac{1}{2\pi}\int_{-\pi}^{\pi}f_\alpha\cdot x_\alpha d\alpha$$

and

$$B_x(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(Q_x^2 BR)_\alpha \cdot \frac{x_\alpha}{|x_\alpha|^2}d\alpha$$

Computing

$$\begin{aligned}\psi_t =& Q_x(Q_x)_t\frac{\gamma}{|x_\alpha|} \underbrace{- \frac{Q_x^2\gamma}{2|x_\alpha|}B_x(t)}_{(1)} - Q_x^2 BR_t\frac{x_\alpha}{|x_\alpha|} - \frac{Q_x^3}{|x_\alpha|}|BR|^2 Q^x_\alpha + Q_x^2 b_s BR_\alpha \cdot \frac{x_\alpha}{|x_\alpha|}\\ &- \frac{Q_x^2}{2|x_\alpha|}\partial_\alpha\left(\frac{\psi^2}{Q_x^2}\right) + \underbrace{\frac{Q_x^2}{2}\left(\frac{b_s^2|x_\alpha|^2}{Q_x^2}\right)_\alpha}_{(1)} - \frac{Q_x^2}{|x_\alpha|}(P_2^{-1}(z))_\alpha - (b_s|x_\alpha|)_t + \mathcal{E}^1\end{aligned}$$

where

$$\mathcal{E}^1 = \frac{Q_x^2}{|x_\alpha|}BR_\alpha \cdot x_\alpha b_e + \frac{Q_x^2}{2|x_\alpha|}(b_e\gamma)_\alpha + \frac{Q_x^2}{|x_\alpha|}g - \frac{Q_x^2\gamma}{2|x_\alpha|}\frac{1}{2\pi}\int_{-\pi}^{\pi} f_\alpha \cdot \frac{x_\alpha}{|x_\alpha|^2}d\alpha$$

are the error terms. We consider

$$\begin{aligned}(1) =& -\frac{Q_x^2\gamma}{2|x_\alpha|}B_x(t) + \frac{Q_x^2}{2}\left(\frac{b_s^2|x_\alpha|^2}{Q_x^2}\right)_\alpha\\ =& -\frac{Q_x^2\gamma}{2|x_\alpha|}B_x(t) + b_s|x_\alpha|(b_s)_\alpha - \frac{Q^x_\alpha}{Q_x}b_s^2|x_\alpha|\\ =& -\frac{Q_x^2\gamma}{2|x_\alpha|}B_x(t) + b_s|x_\alpha|B_x(t) - b_s(Q_x^2 BR)_\alpha \cdot \frac{x_\alpha}{|x_\alpha|} - \frac{Q^x_\alpha}{Q_x}b_s^2|x_\alpha|\\ =& -B_x(t)\psi - b_s(Q_x^2 BR)_\alpha \cdot \frac{x_\alpha}{|x_\alpha|} - \frac{Q^x_\alpha}{Q_x}b_s^2|x_\alpha|\end{aligned}$$

It yields

$$\begin{aligned}\psi_t =& Q_x(Q_x)_t\frac{\gamma}{|x_\alpha|} - B_x(t)\psi \underbrace{- b_s(Q_x^2 BR)_\alpha \cdot \frac{x_\alpha}{|x_\alpha|}}_{(2)} - \frac{Q^x_\alpha}{Q_x}b_s^2|x_\alpha|\\ &- Q_x^2 BR_t\frac{x_\alpha}{|x_\alpha|} - \frac{Q_x^3}{|x_\alpha|}|BR|^2 Q^x_\alpha + \underbrace{Q_x^2 b_s BR_\alpha \cdot \frac{x_\alpha}{|x_\alpha|}}_{(2)}\\ &- \frac{Q_x^2}{2|x_\alpha|}\partial_\alpha\left(\frac{\psi^2}{Q_x^2}\right) - \frac{Q_x^2}{|x_\alpha|}(P_2^{-1}(z))_\alpha - (b_s|x_\alpha|)_t + \mathcal{E}^1\end{aligned}$$

It is easy to check that

$$(2) = -b_s(Q_x^2 BR)_\alpha \cdot \frac{x_\alpha}{|x_\alpha|} + Q_x^2 b_s BR_\alpha \cdot \frac{x_\alpha}{|x_\alpha|} = -2b_s BR \cdot \frac{x_\alpha}{|x_\alpha|}Q_x(Q_x)_\alpha,$$

then

$$\begin{aligned}\psi_t =& - B_x(t)\psi - \frac{Q_x^2}{2|x_\alpha|}\partial_\alpha\left(\frac{\psi^2}{Q_x^2}\right) - Q_x^2\left(BR_t\frac{x_\alpha}{|x_\alpha|} + \frac{(P_2^{-1}(z))_\alpha}{|x_\alpha|}\right)\\ &+ Q_x(Q_x)_t\frac{\gamma}{|x_\alpha|} - 2b_sBR\cdot\frac{x_\alpha}{|x_\alpha|}Q_x(Q_x)_\alpha - \frac{Q_\alpha^x}{Q_x}b_s^2|x_\alpha| - \frac{Q_x^3}{|x_\alpha|}|BR|^2Q_\alpha^x\\ &- (b_s|x_\alpha|)_t + \mathcal{E}^1\end{aligned}$$

With this formula it is easy to find that

$$\frac{1}{2}\frac{d}{dt}\int|\mathcal{D}|^2dx \leq CP(E(t)) + c\delta(t)$$

In order to deal with II

$$II = \int_{-\pi}^{\pi}\Lambda\partial_\alpha^3\mathcal{D}\partial_\alpha^3\mathcal{D}_td\alpha$$

we take a derivative in α in the equation for ω and ψ to reorganize the most dangerous terms. If we find a term of low order, we will denote it by NICE. Since the equations for φ_t and ψ_t are analogous except for the $\mathcal{E}^1$ term, the NICE terms are going to be easier to estimate in terms of $CP(E(t)) + c\delta(t)$.

$$\begin{aligned}\psi_{\alpha t} =& - B_x(t)\psi_\alpha - \partial_\alpha\left(\frac{Q_x^2}{2|x_\alpha|}\partial_\alpha\left(\frac{\psi^2}{Q_x^2}\right)\right) - \left(Q_x^2\left(\underbrace{BR_t\frac{x_\alpha}{|x_\alpha|}}_{(3)} + \frac{(P_2^{-1}(z))_\alpha}{|x_\alpha|}\right)\right)_\alpha\\ &+ \left(Q_x(Q_x)_t\frac{\gamma}{|x_\alpha}|\right)_\alpha - \left(2b_sBR\cdot\frac{x_\alpha}{|x_\alpha|}Q_x(Q_x)_\alpha\right)_\alpha - \left(\frac{Q_\alpha^x}{Q_x}b_s^2|x_\alpha|\right)_\alpha - \left(\frac{Q_x^3}{|x_\alpha|}|BR|^2Q_\alpha^x\right)_\alpha\\ &\underbrace{-(b_s|x_\alpha|)_{\alpha t}}_{(3)} + \mathcal{E}_\alpha^1\end{aligned}$$

Expanding (3):

$$\begin{aligned}(3) =& - \left(Q_x^2BR_t\frac{x_\alpha}{|x_\alpha|}\right)_\alpha - (b_s|x_\alpha|)_{\alpha t}\\ =& - \left(Q_x^2BR_t\right)_\alpha\frac{x_\alpha}{|x_\alpha|} - Q_x^2BR_t\left(\frac{x_\alpha}{|x_\alpha|}\right)_\alpha - \left(|x_\alpha|B_x(t) - (Q_x^2BR)_\alpha\cdot\frac{x_\alpha}{|x_\alpha|}\right)_t\\ =& - \left(|x_\alpha|B_x(t)\right)_t + (Q_x^2BR)_\alpha\cdot\left(\frac{x_\alpha}{|x_\alpha|}\right)_t + 2(Q_x(Q_x)_tBR)_\alpha\cdot\frac{x_\alpha}{|x_\alpha|} - Q_x^2BR_t\cdot\left(\frac{x_\alpha}{|x_\alpha|}\right)_\alpha\end{aligned}$$

We use that

$$\left(\frac{x_\alpha}{|x_\alpha|}\right)_\alpha = \frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^2}\cdot\frac{x_\alpha^\perp}{|x_\alpha|};\quad \left(\frac{x_\alpha}{|x_\alpha|}\right)_t = \frac{x_{\alpha t}\cdot x_\alpha^\perp}{|x_\alpha|^2}\cdot\frac{x_\alpha^\perp}{|x_\alpha|}$$

to find

$$\psi_{\alpha t} = \underbrace{-B_x(t)\psi_\alpha}_{(4)} - \underbrace{\frac{\partial^2_\alpha(\psi^2)}{2|x_\alpha|}}_{(5)} + \underbrace{\partial_\alpha\left(\frac{(Q_x)_\alpha}{|x_\alpha|Q_x}\psi^2\right)}_{(6)} - Q_x^2 BR_t \cdot x_\alpha^\perp \frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3} - (|x_\alpha|B_x(t))_t$$
$$+ \underbrace{(Q_x^2 BR)_\alpha \cdot x_\alpha^\perp \frac{x_{\alpha t}\cdot x_\alpha^\perp}{|x_\alpha|^3}}_{(13)} + \underbrace{2(Q_x(Q_x)_t BR)_\alpha \frac{x_\alpha}{|x_\alpha|}}_{(7)} - \underbrace{\left(Q_x^2\frac{(P_2^{-1}(z))_\alpha}{|x_\alpha|}\right)_\alpha}_{(8)}$$
$$+ \underbrace{\left(Q_x(Q_x)_t\frac{\gamma}{|x_\alpha|}\right)_\alpha}_{(9)} - \underbrace{\left(2b_s BR\cdot\frac{x_\alpha}{|x_\alpha|}Q_x(Q_x)_\alpha\right)_\alpha}_{(10)} - \underbrace{\left(\frac{(Q_x)_\alpha}{Q_x}b_s^2|x_\alpha|\right)_\alpha}_{(11)}$$
$$- \underbrace{\left(\frac{Q_x^3}{|x_\alpha|}|BR|^2(Q_x)_\alpha\right)_\alpha}_{(12)} + \mathcal{E}^1_\alpha$$

The term $(|x_\alpha|B_x(t))_t$ depends only on t so it is not going to appear computing II.

$$(4) = -B_x(t)\psi_\alpha \text{ is NICE (at the level of } \psi_\alpha)$$

$$(5) = -\frac{\partial^2_\alpha(\psi^2)}{2|x_\alpha|} \text{ is a transparent term which is NICE (even if we have to deal with } \Lambda^{1/2})$$

$$(6) = \partial_\alpha\left(\frac{(Q_x)_\alpha}{|x_\alpha|Q_x}\psi^2\right) = -\frac{(Q_x)^2_\alpha}{|x_\alpha|(Q_x)^2} + \frac{2(Q_x)_\alpha\psi\psi_\alpha}{|x_\alpha|Q_x} + \frac{\psi^2}{Q_x}\left(\frac{(Q_x)_\alpha}{|x_\alpha|}\right)_\alpha$$

The first term is at the level of $\partial_\alpha x$ so it is NICE. The second term is at the level of $\partial_\alpha x$ or ψ_α so it is NICE. We write the last one as

$$\frac{\psi^2}{Q_x}\left(\frac{(Q_x)_\alpha}{|x_\alpha|}\right)_\alpha = \frac{\psi^2}{Q_x}x_\alpha\cdot\left(\nabla^2 Q(x)\cdot\frac{x_\alpha}{|x_\alpha|}\right) + \frac{\psi^2}{Q_x}x_\alpha\nabla Q(x)\cdot x_\alpha^\perp\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}$$

The first term is at the level of x_α or ψ so it is NICE. For the second term we have used that

$$\left(\frac{x_\alpha}{|x_\alpha|}\right)_\alpha = \frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^2}\cdot\frac{x_\alpha^\perp}{|x_\alpha|}$$

Finally:

$$(6) = \text{NICE } + \frac{\psi^2}{Q_x}x_\alpha\nabla Q(x)\cdot x_\alpha^\perp\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}$$

$$
\begin{aligned}
(7) = 2(Q_x(Q_x)_tBR)_\alpha \frac{x_\alpha}{|x_\alpha|} = 2(Q_x)_\alpha(Q_x)_tBR \cdot \frac{x_\alpha}{|x_\alpha|} + 2Q_x\left(\frac{(Q_x)_t}{|x_\alpha}\right)_\alpha BR\cdot x_\alpha \\
+ 2Q_x(Q_x)_tBR_\alpha \cdot \frac{x_\alpha}{|x_\alpha|}
\end{aligned}
$$

The first term is at the level of $x_\alpha, x_t, BR \sim x_\alpha$ so it is NICE. We use that

$$
\frac{(Q_x)_{t\alpha}}{|x_\alpha|} = \frac{(Q_x)_{\alpha t}}{|x_\alpha|} = \frac{(\nabla Q(x)\cdot x_\alpha)_t}{|x_\alpha|} = \left(\nabla Q(x)\cdot\frac{x_\alpha}{|x_\alpha|}\right)_t - \nabla Q(x)\cdot x_\alpha\left(\frac{1}{|x_\alpha|}\right)_t
$$

Using that

$$
\frac{x_\alpha\cdot x_{\alpha t}}{|x_\alpha|^2} = B_x(t) + \frac{1}{2\pi}\int_{-\pi}^{\pi} f_\alpha\cdot\frac{x_\alpha}{|x_\alpha|^2}d\alpha
$$

and

$$
\left(\frac{x_\alpha}{|x_\alpha|}\right)_t = \frac{x_{\alpha t}\cdot x_\alpha^\perp}{|x_\alpha|^2}\cdot\frac{x_\alpha^\perp}{|x_\alpha|}
$$

we find that

$$
\begin{aligned}
\frac{(Q_x)_{t\alpha}}{|x_\alpha|} = x_t\cdot\left(\nabla^2 Q(x)\cdot\frac{x_\alpha}{|x_\alpha|}\right) + \nabla Q(x)\cdot x_\alpha^\perp\frac{x_{\alpha t}\cdot x_\alpha^\perp}{|x_\alpha|^3} \\
+ \nabla Q(x)\cdot\frac{x_\alpha}{|x_\alpha|}B_x(t) + \nabla Q(x)\cdot\frac{x_\alpha}{|x_\alpha|}\frac{1}{2\pi}\int_{-\pi}^{\pi} f_\alpha\cdot\frac{x_\alpha}{|x_\alpha|^2}d\alpha
\end{aligned} \tag{5.12}
$$

That yields

$$
\begin{aligned}
(7) = 2(Q_x(Q_x)_tBR)_\alpha\frac{x_\alpha}{|x_\alpha|} = \text{NICE } + \underbrace{2Q_xBR\cdot x_\alpha x_t\cdot\left(\nabla^2Q(x)\cdot\frac{x_\alpha}{|x_\alpha|}\right)}_{\text{NICE (at the level of } x_\alpha, x_t, BR)} \\
+ 2Q_xBR\cdot x_\alpha\nabla Q(x)\cdot x_\alpha^\perp\frac{x_{\alpha t}\cdot x_\alpha^\perp}{|x_\alpha|^3} + \underbrace{2Q_xBR\cdot x_\alpha\nabla Q(x)\cdot\frac{x_\alpha}{|x_\alpha|}B_x(t)}_{\text{NICE (at the level of } x_\alpha, x_t, BR)} \\
+ \underbrace{2Q_xBR\cdot x_\alpha\nabla Q(x)\cdot\frac{x_\alpha}{|x_\alpha|}\frac{1}{2\pi}\int_{-\pi}^{\pi} f_\alpha\cdot\frac{x_\alpha}{|x_\alpha|^2}d\alpha}_{\text{part of error terms}} + 2Q_x(Q_x)_tBR_\alpha\cdot\frac{x_\alpha}{|x_\alpha|}
\end{aligned}
$$

Finally:

$$
(7) = 2(Q_x(Q_x)_tBR)_\alpha\frac{x_\alpha}{|x_\alpha|} = \text{NICE } + 2Q_xBR\cdot x_\alpha\nabla Q(x)\cdot x_\alpha^\perp\frac{x_{\alpha t}\cdot x_\alpha^\perp}{|x_\alpha|^3} + 2Q_x(Q_x)_tBR_\alpha\cdot\frac{x_\alpha}{|x_\alpha|}
$$

$$(8) = -\left(Q_x^2 \frac{(P_2^{-1}(z))_\alpha}{|x_\alpha|}\right)_\alpha = -\left(Q_x^2 \nabla P_2^{-1}(x)\cdot \frac{x_\alpha}{|x_\alpha|}\right)_\alpha = \underbrace{-2Q_x \nabla Q_x^x \cdot x_\alpha \nabla P_2^{-1}(x)\cdot \frac{x_\alpha}{|x_\alpha|}}_{\text{NICE (at the level of } x_\alpha)}$$
$$\underbrace{-Q_x^2 x_\alpha \cdot \left(\nabla^2 P_2^{-1}(x)\cdot \frac{x_\alpha}{|x_\alpha|}\right)}_{\text{NICE (at the level of } x_\alpha)} - Q_x^2 \nabla P_2^{-1}(x)\cdot x_\alpha^\perp \frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}$$

which means

$$(8) = -\left(Q_x^2 \frac{(P_2^{-1}(z))_\alpha}{|x_\alpha|}\right)_\alpha = \text{NICE } - Q_x^2 \nabla P_2^{-1}(x)\cdot x_\alpha^\perp \frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}$$

$$(9) = \left(Q_x (Q_x)_t \frac{\gamma}{|x_\alpha|}\right)_\alpha = \underbrace{(Q_x)_\alpha (Q_x)_t \frac{\gamma}{|x_\alpha|}}_{\text{NICE (at the level of } x_\alpha, x_t)} + Q_x \frac{(Q_x)_{\alpha t}}{|x_\alpha|}\gamma + Q_x (Q_x)_t \left(\frac{\gamma}{|x_\alpha|}\right)_\alpha$$

We use (5.12) to deal with $\frac{(Q_x)_{\alpha t}}{|x_\alpha|}$. We find that

$$(9) = \left(Q_x (Q_x)_t \frac{\gamma}{|x_\alpha|}\right)_\alpha = \text{NICE } + Q_x \gamma \nabla Q(x)\cdot x_\alpha^\perp \frac{x_{\alpha t}\cdot x_\alpha^\perp}{|x_\alpha|^3} + Q_x (Q_x)_t \left(\frac{\gamma}{|x_\alpha|}\right)_\alpha$$

$$(10) = -\left(2b_s BR \cdot \frac{x_\alpha}{|x_\alpha|} Q_x (Q_x)_\alpha\right)_\alpha = \underbrace{-2b_s BR \cdot \frac{x_\alpha}{|x_\alpha|}(Q_x)_\alpha^2}_{\text{NICE as before}} - \left(2b_s BR \cdot \frac{x_\alpha}{|x_\alpha|}\right)_\alpha Q_x (Q_x)_\alpha$$
$$- 2b_s BR \cdot x_\alpha Q_x \nabla Q_x(x)\cdot x_\alpha^\perp \frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3} \underbrace{-2b_s BR \cdot \frac{x_\alpha}{|x_\alpha|} Q_x x_\alpha (\nabla^2 Q_x(x))\cdot x_\alpha}_{\text{NICE as before}}$$

Therefore

$$(10) = -\left(2b_s BR \cdot \frac{x_\alpha}{|x_\alpha|} Q_x (Q_x)_\alpha\right)_\alpha = \text{NICE } - \left(2b_s BR \cdot \frac{x_\alpha}{|x_\alpha|}\right)_\alpha Q_x (Q_x)_\alpha$$
$$- 2b_s BR \cdot x_\alpha Q_x \nabla Q_x(x)\cdot x_\alpha^\perp \frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}$$

$$(11) = -\left(\frac{(Q_x)_\alpha}{Q_x} b_s^2 |x_\alpha|\right)_\alpha = -\left(b_s^2 |x_\alpha|\right)_\alpha \frac{(Q_x)_\alpha}{Q_x} - \frac{b_s^2 |x_\alpha|^2}{Q_x}\nabla Q(x)\cdot x_\alpha^\perp \frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}$$
$$- \frac{x_\alpha (\nabla^2 Q(x)\cdot x_\alpha)}{Q_x} b_s^2 |x_\alpha| + \frac{(Q_x)_\alpha^2}{(Q_x)^2} b_s^2 |x_\alpha|$$

The fact that the last two terms are NICE, allows us to find that

$$(11) = -\left(\frac{(Q_x)_\alpha}{Q_x} b_s^2 |x_\alpha|\right)_\alpha = \text{NICE } - \left(b_s^2|x_\alpha|\right)_\alpha \frac{(Q_x)_\alpha}{Q_x} - \frac{b_s^2|x_\alpha|^2}{Q_x} \nabla Q(x) \cdot x_\alpha^\perp \frac{x_{\alpha\alpha} \cdot x_\alpha^\perp}{|x_\alpha|^3}$$

Finally:

$$\begin{aligned}(12) = -\left(\frac{Q_x^3}{|x_\alpha|}|BR|^2(Q_x)_\alpha\right)_\alpha &= \underbrace{-3(Q_x)^2(Q_x)_\alpha^2|BR|^2}_{\text{NICE}} - \frac{Q_x^3}{|x_\alpha|}(|BR|^2)_\alpha(Q_x)_\alpha \\ &\quad \underbrace{-\frac{Q_x^3}{|x_\alpha|}|BR|^2 x_\alpha \cdot (\nabla^2 Q(x) \cdot x_\alpha)}_{\text{NICE}} - Q_x^3|BR|^2 \nabla Q(x) \cdot x_\alpha^\perp \frac{x_{\alpha\alpha} \cdot x_\alpha^\perp}{|x_\alpha|^3}\end{aligned}$$

which implies that

$$(12) = -\left(\frac{Q_x^3}{|x_\alpha|}|BR|^2(Q_x)_\alpha\right)_\alpha = \text{NICE } - \frac{Q_x^3}{|x_\alpha|}(|BR|^2)_\alpha(Q_x)_\alpha - Q_x^3|BR|^2\nabla Q(x) \cdot x_\alpha^\perp \frac{x_{\alpha\alpha} \cdot x_\alpha^\perp}{|x_\alpha|^3}$$

We gather all the formulas for (4) to (12) using that the error terms will be collected by $\mathcal{E}_\alpha^1$. We will denote the new term by $\tilde{\mathcal{E}}_\alpha^1$.

It yields:

$$\begin{aligned}\psi_{\alpha t} = \text{NICE } &+ \underbrace{\frac{\psi^2}{Q_x}\nabla Q(x) \cdot x_\alpha^\perp \frac{x_{\alpha\alpha} \cdot x_\alpha^\perp}{|x_\alpha|^3}}_{(16)} \underbrace{-Q_x^2 BR_t \cdot x_\alpha^\perp \frac{x_{\alpha\alpha} \cdot x_\alpha^\perp}{|x_\alpha|^3}}_{(15)} \underbrace{-Q_x^2 \nabla P_2^{-1}(x) \cdot x_\alpha^\perp \frac{x_{\alpha\alpha} \cdot x_\alpha^\perp}{|x_\alpha|^3}}_{(15)} \\ &+ \underbrace{Q_x \gamma \nabla Q(x) \cdot x_\alpha^\perp \frac{x_{\alpha t} \cdot x_\alpha^\perp}{|x_\alpha|^3}}_{(18)} + \underbrace{Q_x(Q_x)_t\left(\frac{\gamma}{|x_\alpha|}\right)_\alpha}_{(14)} + \underbrace{2Q_x BR \cdot x_\alpha \nabla Q(x) \cdot x_\alpha^\perp \frac{x_{\alpha t} \cdot x_\alpha^\perp}{|x_\alpha|^3}}_{(18)} \\ &+ \underbrace{Q_x(Q_x)_t 2BR_\alpha \cdot \frac{x_\alpha}{|x_\alpha|}}_{(14)} \underbrace{-\left(2b_x BR \cdot \frac{x_\alpha}{|x_\alpha|}\right)_\alpha Q_x(Q_x)_\alpha}_{(17)} \underbrace{-2b_s BR \cdot x_\alpha Q_x \nabla Q(x) \cdot x_\alpha^\perp \frac{x_{\alpha\alpha} \cdot x_\alpha^\perp}{|x_\alpha|^3}}_{(16)} \\ &\underbrace{-(b_s^2|x_\alpha|)_\alpha \frac{(Q_x)_\alpha}{Q_x}}_{(17)} \underbrace{-\frac{b_s^2|x_\alpha|^2}{Q_x}\nabla Q(x) \cdot x_\alpha^\perp \frac{x_{\alpha\alpha} \cdot x_\alpha^\perp}{|x_\alpha|^3}}_{(16)} \underbrace{-\frac{Q_x^3}{|x_\alpha|}(|BR|^2)_\alpha(Q_x)_\alpha}_{(17)} \\ &\underbrace{-Q_x^3|BR|^2\nabla Q(x) \cdot x_\alpha^\perp \frac{x_{\alpha\alpha} \cdot x_\alpha^\perp}{|x_\alpha|^3}}_{(16)} + (Q_x^2 BR)_\alpha \cdot x_\alpha^\perp \frac{x_{\alpha t} \cdot x_\alpha^\perp}{|x_\alpha|^3} + \tilde{\mathcal{E}}_\alpha^1\end{aligned}$$

We compute

$$
\begin{aligned}
(14) &= Q_x(Q_x)_t\left(\frac{\gamma}{|x_\alpha|}\right)_\alpha + Q_x(Q_x)_t 2BR_\alpha\cdot\frac{x_\alpha}{|x_\alpha|}\\
&= 2\frac{(Q_x)_t}{Q_x}(Q_x)^2\left(\frac{\gamma}{2|x_\alpha|}\right)_\alpha + 2\frac{(Q_x)_t}{Q_x}(Q_x)^2BR_\alpha\cdot\frac{x_\alpha}{|x_\alpha|}\\
&= 2\frac{(Q_x)_t}{Q_x}\psi_\alpha - 2\frac{(Q_x)_t}{Q_x}(Q_x^2)_\alpha\frac{\gamma}{2|x_\alpha|} - 2\frac{(Q_x)_t}{Q_x}(Q_x^2)_\alpha BR_\alpha\frac{x_\alpha}{|x_\alpha|} - 2\frac{(Q_x)_t}{Q_x}(|x_\alpha|B_x(t))
\end{aligned}
$$

The last formula allows us to conclude that (14)=NICE. We reorganize using (15), (16), (17) and (18).

$$
\begin{aligned}
\psi_{\alpha t} &= \text{NICE } - Q_x^2(BR_t\cdot x_\alpha^\perp + \nabla P_2^{-1}(x)\cdot x_\alpha^\perp)\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}\\
&\quad - Q^3\left(|BR|^2 + \frac{b_s^2|x_\alpha|^2}{Q_x^4} + 2b_s\frac{BR\cdot x_\alpha}{Q_x^2} - \frac{\psi^2}{Q_x^4}\right)\nabla Q(x)\cdot x_\alpha^\perp\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}\\
&\quad + (Q_x^2BR)_\alpha\cdot x_\alpha^\perp\frac{x_{\alpha t}\cdot x_\alpha^\perp}{|x_\alpha|^3} + (Q_x\gamma + 2Q_xBR\cdot x_\alpha)\nabla Q(x)\cdot x_\alpha^\perp\frac{x_{\alpha t}\cdot x_\alpha^\perp}{|x_\alpha|^3}\\
&\quad - \left(\frac{Q_x^3(|BR|^2)_\alpha}{|x_\alpha|} + \frac{(b_s^2|x_\alpha|)_\alpha}{Q_x} + \left(2b_sBR\cdot\frac{x_\alpha}{|x_\alpha|}\right)_\alpha Q_x\right)(Q_x)_\alpha + \tilde{\mathcal{E}}_\alpha^1
\end{aligned}
$$

We add and subtract terms in order to find the R-T condition. We remember here that

$$
\begin{aligned}
\sigma_z &= \left(BR_t + \frac{\varphi}{|z_\alpha|}BR_\alpha\right)\cdot z_\alpha^\perp + \frac{\omega}{2|z_\alpha|^2}\left(z_{\alpha t} + \frac{\varphi}{|z_\alpha|}z_{\alpha\alpha}\right)\cdot z_\alpha^\perp\\
&\quad + Q_z\left|BR + \frac{\omega}{2|z_\alpha|^2}z_\alpha\right|^2\nabla Q(z)\cdot z_\alpha^\perp + \nabla P_2^{-1}(z)\cdot z_\alpha^\perp\\
\sigma_x &= \left(BR_t + \frac{\psi}{|x_\alpha|}BR_\alpha\right)\cdot x_\alpha^\perp + \frac{\gamma}{2|x_\alpha|^2}\left(x_{\alpha t} + \frac{\psi}{|x_\alpha|}x_{\alpha\alpha}\right)\cdot x_\alpha^\perp\\
&\quad + Q_x\left|BR + \frac{\gamma}{2|x_\alpha|^2}x_\alpha\right|^2\nabla Q(x)\cdot x_\alpha^\perp + \nabla P_2^{-1}(x)\cdot x_\alpha^\perp
\end{aligned}
\tag{5.13}
$$

In σ_x there are error terms but they are not dangerous. Then, we find

$$
\begin{aligned}
\psi_{\alpha t} &= \text{NICE}\\
&\underbrace{\begin{aligned}&- Q_x^2\left(\left(BR_t + \frac{\psi}{|x_\alpha|}BR_\alpha\right)\cdot x_\alpha^\perp + \frac{\gamma}{2|x_\alpha|^2}\left(x_{\alpha t} + \frac{\psi}{|x_\alpha|}x_{\alpha\alpha}\right)\cdot x_\alpha^\perp + \nabla P_2^{-1}(x)\cdot x_\alpha^\perp\right)\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}\\
&+(Q_x^2BR)_\alpha\cdot x_\alpha^\perp\frac{x_{\alpha t}\cdot x_\alpha^\perp}{|x_\alpha|^3} + Q_x^2\left(\frac{\psi}{|x_\alpha|}BR_\alpha\cdot x_\alpha^\perp + \frac{\gamma}{2|x_\alpha|^2}\left(x_{\alpha t} + \frac{\psi}{|x_\alpha|}x_{\alpha\alpha}\right)\cdot x_\alpha^\perp\right)\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}\end{aligned}}_{(19)}\\
&- Q^3\left(|BR|^2 + \frac{b_s^2|x_\alpha|^2}{Q_x^4} + 2b_s\frac{BR\cdot x_\alpha}{Q_x^2} - \frac{\psi^2}{Q_x^4}\right)\nabla Q(x)\cdot x_\alpha^\perp\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}
\end{aligned}
$$

$$+ (Q_x\gamma + 2Q_xBR\cdot x_\alpha)\nabla Q(x)\cdot x_\alpha^\perp \frac{x_{\alpha t}\cdot x_\alpha^\perp}{|x_\alpha|^3}$$
$$- \left(\frac{Q_x^3(|BR|^2)_\alpha}{|x_\alpha|} + \frac{(b_s^2|x_\alpha|)_\alpha}{Q_x} + \left(2b_sBR\cdot\frac{x_\alpha}{|x_\alpha|}\right)_\alpha Q_x\right)(Q_x)_\alpha + \tilde{\mathcal{E}}_\alpha^1$$

Line (19) can be written as

$$\begin{aligned}
(19) &= (Q_x^2BR)_\alpha\cdot x_\alpha^\perp \frac{x_{\alpha t}\cdot x_\alpha^\perp}{|x_\alpha|^3} + Q_x^2BR_\alpha\cdot x_\alpha^\perp \frac{\psi}{|x_\alpha|}\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}\\
&+ \frac{Q_x^2\gamma}{2|x_\alpha|^2}\left(x_{\alpha t} + \frac{\psi}{|x_\alpha|}x_{\alpha\alpha}\right)\cdot x_\alpha^\perp\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}\\
&= (Q_x^2BR)_\alpha\cdot x_\alpha^\perp \frac{x_{\alpha t}\cdot x_\alpha^\perp}{|x_\alpha|^3} + (Q_x^2BR)_\alpha\cdot x_\alpha^\perp \frac{\psi}{|x_\alpha|}\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}\\
&+ \frac{Q_x^2\gamma}{2|x_\alpha|^2}\left(x_{\alpha t}\cdot x_\alpha^\perp + \frac{\psi}{|x_\alpha|}x_{\alpha\alpha}\cdot x_\alpha^\perp\right)\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3} - 2Q_x(Q_x)_\alpha BR\cdot x_\alpha^\perp\frac{\psi}{|x_\alpha|}\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}\\
&= (Q_x^2BR)_\alpha\cdot x_\alpha^\perp\frac{1}{|x_\alpha|^3}\left(x_{\alpha t}\cdot x_\alpha^\perp + \frac{\psi}{|x_\alpha|}x_{\alpha\alpha}\cdot x_\alpha^\perp\right)\\
&+ \frac{Q_x^2\gamma}{2|x_\alpha|^2}\frac{1}{|x_\alpha|^3}\left(x_{\alpha t}\cdot x_\alpha^\perp + \frac{\psi}{|x_\alpha|}x_{\alpha\alpha}\cdot x_\alpha^\perp\right)x_{\alpha\alpha}\cdot x_\alpha^\perp - 2Q_x(Q_x)_\alpha BR\cdot x_\alpha^\perp\frac{\psi}{|x_\alpha|}\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}\\
&= \frac{1}{|x_\alpha|^3}\left(x_{\alpha t}\cdot x_\alpha^\perp + \frac{\psi}{|x_\alpha|}x_{\alpha\alpha}\cdot x_\alpha^\perp\right)\left((Q_x^2BR)_\alpha\cdot x_\alpha^\perp + \frac{Q_x^2\gamma}{2|x_\alpha|^2}x_{\alpha\alpha}\cdot x_\alpha^\perp\right)\\
&- 2Q_x(Q_x)_\alpha BR\cdot x_\alpha^\perp\frac{\psi}{|x_\alpha|}\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}
\end{aligned}$$

We expand $x_{\alpha t}$ to find

$$\begin{aligned}
(19) &= \frac{1}{|x_\alpha|^3}\left((Q_x^2BR)_\alpha\cdot x_\alpha^\perp + \frac{Q_x^2\gamma}{2|x_\alpha|^2}x_{\alpha\alpha}\cdot x_\alpha^\perp\right)^2\\
&+ \underbrace{\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}\left((Q_x^2BR)_\alpha\cdot x_\alpha^\perp + \frac{Q_x^2\gamma}{2|x_\alpha|^2}x_{\alpha\alpha}\cdot x_\alpha^\perp\right)b_e}_{\text{error term: we incorporate it as } \tilde{\mathcal{E}}_\alpha^2} - 2Q_x(Q_x)_\alpha BR\cdot x_\alpha^\perp\frac{\psi}{|x_\alpha|}\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}
\end{aligned}$$

We denote

$$G_x(\alpha) = (Q_x^2BR)_\alpha\cdot x_\alpha^\perp + \frac{Q_x^2\gamma}{2|x_\alpha|^2}x_{\alpha\alpha}\cdot x_\alpha^\perp \tag{5.14}$$

We claim that

$$G_x(\alpha) = \text{NICE } + |x_\alpha|H(\partial_\alpha\psi)$$

that becomes

$$(G_x(\alpha))^2 = \text{NICE}$$

Then

$$(19) = \text{NICE } -2Q_x(Q_x)_\alpha BR\cdot x_\alpha^\perp \frac{\psi}{|x_\alpha|}\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3} + \tilde{\mathcal{E}}_\alpha^2$$

We write

$$G_x(\alpha) = \underbrace{2Q_x(Q_x)_\alpha BR\cdot x_\alpha^\perp}_{\text{NICE, at the level of } x_\alpha} + \underbrace{Q_x^2\frac{1}{2\pi}\int\frac{(x_\alpha(\alpha)-x_\alpha(\alpha-\beta))\cdot x_\alpha(\alpha)}{|x(\alpha)-x(\alpha-\beta)|^2}\gamma(\alpha-\beta)d\beta}_{\text{NICE, we use that } |x_\alpha|^2=A_x(t)}$$

$$\underbrace{-Q_x^2\frac{1}{\pi}\int\frac{(x_\alpha(\alpha)-x_\alpha(\alpha-\beta))\cdot x_\alpha(\alpha)}{|x(\alpha)-x(\alpha-\beta)|^4}(x(\alpha)-x(\alpha-\beta))(x_\alpha(\alpha)-x_\alpha(\alpha-\beta))\gamma(\alpha-\beta)d\beta}_{\text{NICE, we use that } |x_\alpha|^2 \text{ only depends on time}}$$

$$+ \underbrace{Q_x^2 BR(x,\gamma_\alpha)\cdot x_\alpha^\perp}_{\text{Hilbert transform applied to } \gamma_\alpha} + \frac{Q_x^2\gamma}{2|x_\alpha|^2}x_{\alpha\alpha}\cdot x_\alpha^\perp$$

Therefore

$$\begin{aligned}
G_x(\alpha) &= \text{NICE } + |x_\alpha|Q_x^2 H\left(\left(\frac{\gamma}{2|x_\alpha|}\right)_\alpha\right) + \frac{Q_x^2\gamma}{2|x_\alpha|^2}x_{\alpha\alpha}\cdot x_\alpha^\perp \\
&= \text{NICE } + |x_\alpha| H\left(\left(\frac{Q_x^2\gamma}{2|x_\alpha|}\right)_\alpha\right) + \frac{Q_x^2\gamma}{2|x_\alpha|^2}x_{\alpha\alpha}\cdot x_\alpha^\perp \\
&= \text{NICE } + |x_\alpha| H(\partial_\alpha\psi) + H\left((b_s|x_\alpha|^2)_\alpha\right) + \frac{Q_x^2\gamma}{2|x_\alpha|^2}x_{\alpha\alpha}\cdot x_\alpha^\perp \\
&= \text{NICE } + |x_\alpha| H(\psi_\alpha) - H\left((Q_x^2 BR)_\alpha\cdot x_\alpha\right) + \frac{Q_x^2\gamma}{2|x_\alpha|^2}x_{\alpha\alpha}\cdot x_\alpha^\perp
\end{aligned}$$

$$(Q_x^2BR)_\alpha\cdot x_\alpha = \underbrace{2Q_x(Q_x)_\alpha BR\cdot x_\alpha}_{\text{NICE}} + Q_x^2\frac{1}{2\pi}\int\frac{(x_\alpha(\alpha)-x_\alpha(\alpha-\beta))^\perp\cdot x_\alpha(\alpha)}{|x(\alpha)-x(\alpha-\beta)|^2}\gamma(\alpha-\beta)d\beta$$

$$= \underbrace{-Q_x^2\frac{1}{\pi}\int\frac{(x(\alpha)-x(\alpha-\beta))^\perp\cdot x_\alpha(\alpha)}{|x(\alpha)-x(\alpha-\beta)|^4}(x(\alpha)-x(\alpha-\beta))(x_\alpha(\alpha)-x_\alpha(\alpha-\beta))\gamma(\alpha-\beta)d\beta}_{\text{NICE, extra cancelation in } (x(\alpha)-x(\alpha-\beta))^\perp\cdot x_\alpha(\alpha)}$$

$$+ \underbrace{Q_x^2\frac{1}{2\pi}\int\frac{(x(\alpha)-x(\alpha-\beta))^\perp\cdot x_\alpha(\alpha)}{|x(\alpha)-x(\alpha-\beta)|^2}\gamma(\alpha-\beta)d\beta}_{\text{NICE, extra cancelation in } (x(\alpha)-x(\alpha-\beta))^\perp\cdot x_\alpha(\alpha)}$$

This means that

$$(Q_x^2BR)_\alpha\cdot x_\alpha = \text{NICE } + \frac{1}{2}H\left(Q_x^2\frac{\partial_\alpha^2 x^\perp\cdot x_\alpha}{|x_\alpha|^2}\gamma\right)$$

Taking Hilbert transforms:

$$-H\left((Q_x^2 BR)_\alpha \cdot x_\alpha\right) = \text{NICE } -\frac{1}{2}H^2\left(Q_x^2\frac{\partial_\alpha^2 x^\perp \cdot x_\alpha}{|x_\alpha|^2}\gamma\right) = \text{NICE } +\frac{1}{2}Q_x^2\frac{\partial_\alpha^2 x^\perp \cdot x_\alpha}{|x_\alpha|^2}\gamma$$

Using that $\partial_\alpha^2 x^\perp \cdot x_\alpha = -\partial_\alpha^2 x \cdot x_\alpha^\perp$ we are done. Thus (19) yields

$$\begin{aligned}
\psi_{\alpha t} = \text{NICE } &- Q_x^2\left(\left(BR_t + \frac{\psi}{|x_\alpha|}BR_\alpha\right)\cdot x_\alpha^\perp\right.\\
&+\left.\frac{\gamma}{2|x_\alpha|^2}\left(x_{\alpha t}+\frac{\psi}{|x_\alpha|}x_{\alpha\alpha}\right)\cdot x_\alpha^\perp + \nabla P_2^{-1}(x)\cdot x_\alpha^\perp\right)\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}\\
&- Q_x^3\left(|BR|^2 + \frac{b_s^2|x_\alpha|^2}{Q_x^4} + 2b_s\frac{BR\cdot x_\alpha}{Q_x^2} - \frac{\psi^2}{Q_x^4}\right)\nabla Q(x)\cdot x_\alpha^\perp\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}\\
&+ (Q_x\gamma + 2Q_x BR\cdot x_\alpha)\nabla Q(x)\cdot x_\alpha^\perp\frac{x_{\alpha t}\cdot x_\alpha^\perp}{|x_\alpha|^3}\\
&\underbrace{-\left(\frac{Q_x^3(|BR|^2)_\alpha}{|x_\alpha|} + \frac{(b_s^2|x_\alpha|)_\alpha}{Q_x} + \left(2b_s BR\cdot\frac{x_\alpha}{|x_\alpha|}\right)_\alpha Q_x\right)(Q_x)_\alpha}_{(20)}\\
&\underbrace{-2Q_x(Q_x)_\alpha BR\cdot x_\alpha^\perp\frac{\psi}{|x_\alpha|}\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}}_{(21)} + \mathcal{E}_\alpha^2, \quad \text{where} \quad \mathcal{E}_\alpha^2 = \tilde{\mathcal{E}}_\alpha^1 + \tilde{\mathcal{E}}_\alpha^2
\end{aligned}$$

For (20) we write

$$\begin{aligned}
|x_t|^2 &= Q_x^4|BR|^2 + b_s^2|x_\alpha|^2 + 2Q_x^2 b_s BR\cdot x_\alpha\\
&+ \underbrace{b_e^2|x_\alpha|^2 + f^2 + 2Q_x^2 BR\cdot x_\alpha b_e + 2b_s b_e|x_\alpha|^2 + 2Q_x^2 BR\cdot f + 2b_s x_\alpha\cdot f + 2b_e x_\alpha\cdot f}_{\text{error terms } \tilde{\mathcal{E}}_\alpha^3}
\end{aligned}$$

$$\Rightarrow \frac{|x_t|^2}{Q_x|x_\alpha|} = \frac{Q_x^3|BR|^2}{|x_\alpha|} + \frac{b_s^2|x_\alpha|}{Q_x} + 2Q_x b_s BR\cdot\frac{x_\alpha}{|x_\alpha|} + \frac{\tilde{\mathcal{E}}_\alpha^3}{Q_x|x_\alpha|}$$

Now

$$(20) = \text{NICE } -\frac{(|x_t|^2)_\alpha}{Q_x|x_\alpha|}(Q_x)_\alpha + \frac{\tilde{\mathcal{E}}_\alpha^3}{Q_x|x_\alpha|}(Q_x)_\alpha$$

which means

$$(20)+(21) = \text{NICE } -\frac{(|x_t|^2)_\alpha}{Q_x|x_\alpha|}(Q_x)_\alpha - 2Q_x(Q_x)_\alpha BR\cdot x_\alpha^\perp\frac{\psi}{|x_\alpha|}\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3} + \frac{\tilde{\mathcal{E}}_\alpha^3}{Q_x|x_\alpha|}(Q_x)_\alpha$$

We write

$$
\begin{aligned}
x_{\alpha t} &= \underbrace{(x_{\alpha t}\cdot x_\alpha)\frac{x_\alpha}{|x_\alpha|^2}}_{\text{only depends on } t} + (x_{\alpha t}\cdot x_\alpha^\perp)\frac{x_\alpha^\perp}{|x_\alpha|^2}\\
&= \left(B_x(t) + \frac{1}{2\pi}\int_{-\pi}^{\pi} f_\beta\cdot\frac{x_\beta}{|x_\beta|^2}d\beta\right)x_\alpha + \left((Q_x^2BR)_\alpha\cdot x_\alpha^\perp + bx_{\alpha\alpha}\cdot x_\alpha^\perp + f_\alpha\cdot x_\alpha^\perp\right)\frac{x_\alpha^\perp}{|x_\alpha|^2}\\
&= \left(B_x(t) + \frac{1}{2\pi}\int_{-\pi}^{\pi} f_\beta\cdot\frac{x_\beta}{|x_\beta|^2}d\beta\right)x_\alpha + \left((Q_x^2BR)_\alpha\cdot x_\alpha^\perp + b_sx_{\alpha\alpha}\cdot x_\alpha^\perp\right)\frac{x_\alpha^\perp}{|x_\alpha|^2}\\
&+ \left(b_ex_{\alpha\alpha}\cdot x_\alpha^\perp + f_\alpha\cdot x_\alpha^\perp\right)\frac{x_\alpha^\perp}{|x_\alpha|^2}\\
&= \left(B_x(t) + \frac{1}{2\pi}\int_{-\pi}^{\pi} f_\beta\cdot\frac{x_\beta}{|x_\beta|^2}d\beta\right)x_\alpha + \underbrace{\left((Q_x^2BR)_\alpha\cdot x_\alpha^\perp + \frac{Q_x^2\gamma}{2|x_\alpha|^2}x_{\alpha\alpha}\cdot x_\alpha^\perp\right)}_{G_x(\alpha)\text{ as in (5.14)}}\frac{x_\alpha^\perp}{|x_\alpha|^2}\\
&- \frac{\psi}{|x_\alpha|}x_{\alpha\alpha}\cdot x_\alpha^\perp\frac{x_\alpha^\perp}{|x_\alpha|^2}\left(b_ex_{\alpha\alpha}\cdot x_\alpha^\perp + f_\alpha\cdot x_\alpha^\perp\right)\frac{x_\alpha^\perp}{|x_\alpha|^2}
\end{aligned}
$$

Writing $x_t = (Q_x^2BR) + b_sx_\alpha + b_ex_\alpha + f_\alpha$ we compute

$$
\begin{aligned}
x_{\alpha t}\cdot x_\alpha = \underbrace{Q_x^2BR\cdot x_\alpha}_{\text{NICE}}&\left(B_x(t) + \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} f_\beta\cdot\frac{x_\beta}{|x_\beta|^2}d\beta}_{\text{error}}\right) + \underbrace{G_x(\alpha)Q_x^2BR\cdot\frac{x_\alpha^\perp}{|x_\alpha|^2}}_{\text{NICE because } G_x \text{ is nice}}\\
&- \frac{\psi}{|x_\alpha|}x_{\alpha\alpha}\cdot x_\alpha^\perp Q_x^2BR\cdot\frac{x_\alpha^\perp}{|x_\alpha|^2} + Q_x^2BR\cdot\frac{x_\alpha^\perp}{|x_\alpha|^2}\left(b_ex_{\alpha\alpha}\cdot x_\alpha^\perp + \underbrace{f_\alpha\cdot x_\alpha^\perp}_{\text{error}}\right)\frac{x_\alpha^\perp}{|x_\alpha|^2}\\
&+ \underbrace{b_s\left(B_x(t) + \frac{1}{2\pi}\int_{-\pi}^{\pi} f_\beta\cdot\frac{x_\beta}{|x_\beta|^2}d\beta\right)|x_\alpha|^2}_{\text{NICE}} + b_e\left(\underbrace{B_x(t)}_{\text{error}} + \frac{1}{2\pi}\int_{-\pi}^{\pi} f_\beta\cdot\frac{x_\beta}{|x_\beta|^2}d\beta\right)|x_\alpha|^2 + \hat{\mathcal{E}}
\end{aligned}
$$

where $\hat{\mathcal{E}}$ is an error term. To simplify we write

$$
x_{\alpha t}\cdot x_\alpha = \text{NICE } - \frac{\psi}{|x_\alpha|}x_{\alpha\alpha}\cdot x_\alpha^\perp Q_x^2BR\cdot\frac{x_\alpha^\perp}{|x_\alpha|^2} + \text{ errors}
$$

Setting the above formula in the expression of (20)+(21) allows us to find

$$
(20) + (21) = \text{NICE } + \text{ errors}
$$

This yields

$$\begin{aligned}\psi_{\alpha t} = \text{NICE } &- Q_x^2\left(\left(BR_t + \frac{\psi}{|x_\alpha|}BR_\alpha\right) + \frac{\gamma}{2|x_\alpha|^2}\left(x_{\alpha t} + \frac{\psi}{|x_\alpha|}x_{\alpha\alpha}\right) + \nabla P_2^{-1}(x)\right)\cdot x_\alpha^\perp \frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}\\ &- Q_x^3\left(|BR|^2 + \frac{b_s^2|x_\alpha|^2}{Q_x^4} + 2b_s\frac{BR\cdot x_\alpha}{Q_x^2} - \frac{\psi^2}{Q_x^4}\right)\nabla Q(x)\cdot x_\alpha^\perp\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}\\ &+ (Q_x\gamma + 2Q_xBR\cdot x_\alpha)\nabla Q(x)\cdot x_\alpha^\perp\frac{x_{\alpha t}\cdot x_\alpha^\perp}{|x_\alpha|^3} + \mathcal{E}_\alpha^3\end{aligned}$$

being $\mathcal{E}_\alpha^3$ a new error term. We now complete the formula for σ_x in (5.13) to find

$$\begin{aligned}\psi_{\alpha t} = \text{NICE } &- Q_x^2\sigma_x\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}\\ &+ \underbrace{Q_x\left|BR + \frac{\gamma}{2|x_\alpha|^2}x_\alpha\right|^2\nabla Q(x)\cdot x_\alpha^\perp\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}}_{(22)}\\ &+ \underbrace{Q_x^3\left(-|BR|^2 - \frac{b_s^2|x_\alpha|^2}{Q_x^4} - 2b_s\frac{BR\cdot x_\alpha}{Q_x^2} + \frac{\psi^2}{Q_x^4}\right)\nabla Q(x)\cdot x_\alpha^\perp\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}}_{(23)}\\ &+ \underbrace{(Q_x\gamma + 2Q_xBR\cdot x_\alpha)\nabla Q(x)\cdot x_\alpha^\perp\frac{x_{\alpha t}\cdot x_\alpha^\perp}{|x_\alpha|^3}}_{(24)} + \mathcal{E}_\alpha^3\end{aligned}$$

Expanding

$$\frac{\psi^2}{Q_x^4} = \frac{\gamma^2}{4|x_\alpha|^2} + \frac{b_s^2|x_\alpha|^2}{Q_x^4} - \frac{\gamma b_s}{Q_x^2}$$

we find

$$(22) + (23) = Q_x^3\left(\frac{\gamma^2}{2|x_\alpha|^2} + BR\cdot x_\alpha\frac{\gamma}{|x_\alpha|^2} - 2b_s\frac{BR\cdot x_\alpha}{Q_x^2} - \frac{\gamma b_s}{Q_x^2}\right)\nabla Q(x)\cdot x_\alpha^\perp\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}$$

Writing

$$x_{\alpha t}\cdot x_\alpha^\perp = (Q_x^2BR)_\alpha x_\alpha^\perp + b_s x_{\alpha\alpha}\cdot x_\alpha^\perp + \text{ errors}$$

we obtain that

$$\begin{aligned}(24) = &\,(Q_x\gamma + 2Q_xBR\cdot x_\alpha)\,\nabla Q(x)\cdot x_\alpha^\perp\frac{(Q_x^2BR)_\alpha\cdot x_\alpha^\perp}{|x_\alpha|^3}\\ &+ (Q_x\gamma + 2Q_xBR\cdot x_\alpha)\,\nabla Q(x)\cdot x_\alpha^\perp b_s\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3} + \text{ errors}\end{aligned}$$

Thus

$$
\begin{aligned}
(22)+(23)+(24) &= Q_x^3\left(\frac{\gamma^2}{2|x_\alpha|^2}+BR\cdot x_\alpha\frac{\gamma}{|x_\alpha|^2}\right)\nabla Q(x)\cdot x_\alpha^\perp\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}\\
&\quad+\left(Q_x\gamma+2Q_xBR\cdot x_\alpha\right)\nabla Q(x)\cdot x_\alpha^\perp\frac{(Q_x^2BR)_\alpha\cdot x_\alpha^\perp}{|x_\alpha|^3}+\text{ errors}\\
&= Q_x\nabla Q(x)\cdot x_\alpha^\perp\left(\gamma+2BR\cdot x_\alpha\right)\left(\frac{Q_x^2\gamma}{2|x_\alpha|^2}\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}+\frac{(Q_x^2BR)_\alpha\cdot x_\alpha^\perp}{|x_\alpha|^3}\right)+\text{ errors}\\
&= Q_x\nabla Q(x)\cdot x_\alpha^\perp\left(\gamma+2BR\cdot x_\alpha\right)\frac{1}{|x_\alpha|^3}D_x(\alpha)+\text{ errors}\\
&= \text{NICE }+\text{ errors}
\end{aligned}
$$

Finally, we obtain

$$
\psi_{\alpha t}=\text{NICE}(x,\gamma,\psi)-Q_x^2\sigma_x\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}+\mathcal{E}_\alpha^4
$$

For $\varphi_{\alpha t}$ we find

$$
\varphi_{\alpha t}=\text{NICE}(z,\omega,\varphi)-Q_z^2\sigma_z\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3},
$$

since we can apply the same methods as before to the equations with $f=g=0$, which are satisfied by (z,ω,φ). Then:

$$
\begin{aligned}
II &= \int_{-\pi}^{\pi}\Lambda\partial_\alpha^3\mathcal{D}\cdot\partial_\alpha^3\mathcal{D}_t=\int_{-\pi}^{\pi}\Lambda\partial_\alpha^3\mathcal{D}\left(\text{NICE}(z,\omega,\varphi)-\text{NICE}(x,\gamma,\psi)\right)d\alpha\\
&-\int_{-\pi}^{\pi}\Lambda\partial_\alpha^3\mathcal{D}\left(\partial_\alpha^2\left(Q_z^2\sigma_z\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}-Q_x^2\sigma_x\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}\right)\right)-\int_{-\pi}^{\pi}\Lambda\partial_\alpha^3\mathcal{D}\mathcal{E}_\alpha^4d\alpha\equiv II_1+II_2+II_3
\end{aligned}
$$

$$
II_1\le CP(E(t))\quad\text{because we are dealing with the NICE term}
$$

$$
II_3\le CP(E(t))+c\delta(t)\quad\text{because of the errors}
$$

It remains to estimate II_2. We consider the most singular terms

$$
\begin{aligned}
II_2 &= II_{2,1}+II_{2,2}+II_{2,3}+\text{ l.o.t}\\
II_{2,1} &= -\int_{-\pi}^{\pi}\Lambda(\partial_\alpha^3\mathcal{D})\left((Q_z^2)_{\alpha\alpha}\sigma_z\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}-(Q_x^2)_{\alpha\alpha}\sigma_x\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}\right)d\alpha\\
II_{2,2} &= -\int_{-\pi}^{\pi}\Lambda(\partial_\alpha^3\mathcal{D})\left(Q_z^2\sigma_z\frac{\partial_\alpha^4z\cdot z_\alpha^\perp}{|z_\alpha|^3}-Q_x^2\sigma_x\frac{\partial_\alpha^4x\cdot x_\alpha^\perp}{|x_\alpha|^3}\right)d\alpha\\
II_{2,3} &= -\int_{-\pi}^{\pi}\Lambda(\partial_\alpha^3\mathcal{D})\left(Q_z^2\partial_\alpha^2\sigma_z\frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3}-Q_x^2\partial_\alpha^2\sigma_x\frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3}\right)d\alpha
\end{aligned}
$$

$$
\begin{aligned}
II_{2,1} &= -\int_{-\pi}^{\pi} H(\partial_\alpha^3 \mathcal{D}) \left((Q_z^2)_{\alpha\alpha} \sigma_z \frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3} - (Q_x^2)_{\alpha\alpha}\sigma_x \frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3} \right)_\alpha d\alpha \\
&\leq CP(E(t)) + c\delta(t) \quad \text{ as before}
\end{aligned}
$$

For $II_{2,2}$ we decompose further

$$
\begin{aligned}
II_{2,2} &= -S + \widetilde{II}_{2,2}, \quad \text{ where} \\
\widetilde{II}_{2,2} &= -\int_{-\pi}^{\pi} \Lambda(\partial_\alpha^3 \mathcal{D}) \left(Q_z^2 \sigma_z \frac{\partial_\alpha^4 x \cdot z_\alpha^\perp}{|z_\alpha|^3} - Q_x^2 \sigma_x \frac{\partial_\alpha^4 x \cdot x_\alpha^\perp}{|x_\alpha|^3} \right) d\alpha \\
S &= \int_{-\pi}^{\pi} Q_z^2 \sigma_z \partial_\alpha^4 D \cdot \frac{\partial_\alpha z^\perp(\alpha)}{|z_\alpha|^3} H(\partial_\alpha^4 \mathcal{D})(\alpha) d\alpha
\end{aligned}
$$

We find that

$$
\widetilde{II}_{2,2} \leq CP(E(t)) + c\delta(t)
$$

and $-S$ cancels out with S. We are done with $II_{2,2}$. We write

$$
II_{2,3} = \int_{-\pi}^{\pi} H(\partial_\alpha^3 \mathcal{D}) \left(Q_z^2 \partial_\alpha^3 \sigma_z \frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3} - Q_x^2 \partial_\alpha^3 \sigma_x \frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3} \right) d\alpha + \quad \text{l.o.t}
$$

We claim that

$$
Q_x^2 \partial_\alpha^3 \sigma_x = |x_\alpha| H(\partial_\alpha^3 \psi_t) - b_s |x_\alpha| H(\partial_\alpha^4 \psi) + \text{ errors } + \text{NICE}(x, \gamma, \psi) \tag{5.15}
$$

In the local existence we get

$$
Q_z^2 \partial_\alpha^3 \sigma_z = |z_\alpha| H(\partial_\alpha^3 \varphi_t) - c|z_\alpha| H(\partial_\alpha^4 \varphi) + \text{NICE}(z, \omega, \varphi)
$$

This implies

$$
\begin{aligned}
II_{2,3} &= II_{2,3,1} + II_{2,3,2} + II_{2,3,3} + II_{2,3,4} \\
II_{2,3,1} &= \int_{-\pi}^{\pi} H(\partial_\alpha^3 \mathcal{D}) \left(|z_\alpha| H(\partial_\alpha^3 \varphi_t) \frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3} - |x_\alpha| H(\partial_\alpha^3 \psi_t) \frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3} \right) d\alpha \\
II_{2,3,2} &= -\int_{-\pi}^{\pi} H(\partial_\alpha^3 \mathcal{D}) \left(c|z_\alpha| H(\partial_\alpha^4 \varphi) \frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3} - b_s |x_\alpha| H(\partial_\alpha^4 \psi) \frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3} \right) d\alpha \\
II_{2,3,3} &= \int_{-\pi}^{\pi} H(\partial_\alpha^3 \mathcal{D}) \left(\text{NICE}(z, \omega, \varphi) \frac{z_{\alpha\alpha}\cdot z_\alpha^\perp}{|z_\alpha|^3} - \text{NICE}(x, \gamma, \psi) \frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3} \right) d\alpha \\
II_{2,3,4} &= -\int_{-\pi}^{\pi} H(\partial_\alpha^3 \mathcal{D}) \frac{x_{\alpha\alpha}\cdot x_\alpha^\perp}{|x_\alpha|^3} d\alpha + \text{ errors}
\end{aligned}
$$

It is easy to find

$$
II_{2,3,4} \leq CP(E(t)) + c\delta(t), \quad \text{ error terms}
$$

$$II_{2,3,3} \leq CP(E(t)), \qquad \text{l.o.t}$$

In $II_{2,3,2}$ we split further:

$$\begin{aligned}
II_{2,3,2} &= II^1_{2,3,2} + II^2_{2,3,2}\\
II^1_{2,3,2} &= -\int_{-\pi}^{\pi} H(\partial^3_\alpha \mathcal{D})|z_\alpha| H(\partial^4_\alpha \mathcal{D}) \frac{z_{\alpha\alpha}\cdot z^\perp_\alpha}{|z_\alpha|^3} d\alpha\\
II^2_{2,3,2} &= -\int_{-\pi}^{\pi} H(\partial^3_\alpha \mathcal{D}) H(\partial^4_\alpha \psi) \left(c|z_\alpha| \frac{z_{\alpha\alpha}\cdot z^\perp_\alpha}{|z_\alpha|^3} - b_s |x_\alpha| \frac{x_{\alpha\alpha}\cdot x^\perp_\alpha}{|x_\alpha|^3} \right) d\alpha
\end{aligned}$$

Then

$$II^1_{2,3,2} = \frac{1}{2}\int_{-\pi}^{\pi} |H(\partial^3_\alpha \mathcal{D})|^2 \left(|z_\alpha| \frac{z_{\alpha\alpha}\cdot z^\perp_\alpha}{|z_\alpha|^3} \right)_\alpha d\alpha \leq CP(E(t))$$

For $II^2_{2,3,2}$:

$$II^2_{2,3,2} = -\int_{-\pi}^{\pi} \Lambda^{1/2}(\partial^3_\alpha \psi) \Lambda^{1/2} \left(H(\partial^3_\alpha \mathcal{D}) \left(c|z_\alpha| \frac{z_{\alpha\alpha}\cdot z^\perp_\alpha}{|z_\alpha|^3} - b_s |x_\alpha| \frac{x_{\alpha\alpha}\cdot x^\perp_\alpha}{|x_\alpha|^3} \right) \right) d\alpha \leq CP(E(t))$$

It remains

$$\begin{aligned}
II_{2,3,1} &= II^1_{2,3,1} + II^2_{2,3,1}\\
II^1_{2,3,1} &= \int_{-\pi}^{\pi} H(\partial^3_\alpha \mathcal{D}) H(\partial^3_\alpha \mathcal{D}_t) \frac{z_{\alpha\alpha}\cdot z^\perp_\alpha}{|z_\alpha|^2} d\alpha\\
II^2_{2,3,1} &= \int_{-\pi}^{\pi} H(\partial^3_\alpha \mathcal{D}) \underbrace{H(\partial^3_\alpha \psi_t)}_{\text{approx. sol.}} \left(\frac{z_{\alpha\alpha}\cdot z^\perp_\alpha}{|z_\alpha|^2} - \frac{x_{\alpha\alpha}\cdot x^\perp_\alpha}{|x_\alpha|^2} \right) d\alpha
\end{aligned}$$

Then

$$II^1_{2,3,1} \leq CP(E(t)) + c\delta(t)$$

At this point we remember that we had to deal with

$$II = \int_{-\pi}^{\pi} \Lambda(\partial^3_\alpha \mathcal{D}) \partial^3_\alpha \mathcal{D}_t d\alpha$$

so in $II^1_{2,3,1}$ we find one derivative less (or 1/2 derivatives less) and this shows that we can bound

$$II^1_{2,3,1} \leq CP(E(t)) + c\delta(t)$$

by brute force. It remains to show claim (5.15). We remember

$$Q_x^2\sigma_x = Q_x^2\left(BR_t + \frac{\psi}{|x_\alpha|}BR_\alpha\right)\cdot x_\alpha^\perp + \frac{Q_x^2\gamma}{2|x_\alpha|^2}\left(x_{\alpha t} + \frac{\psi}{|x_\alpha|}x_{\alpha\alpha}\right)\cdot x_\alpha^\perp$$
$$+ \underbrace{Q_x^3\left|BR + \frac{\gamma}{2|x_\alpha|^2}x_\alpha\right|^2\nabla Q(x)\cdot x_\alpha^\perp}_{\text{this term is in } H^3 \text{ so it is NICE}} + \underbrace{Q_x^2\nabla P_2^{-1}(x)\cdot x_\alpha^\perp}_{\text{this term is also in } H^3}$$

We write

$$\frac{Q_x^2\gamma}{2|x_\alpha|^2}\left(x_{\alpha t} + \frac{\psi}{|x_\alpha|}x_{\alpha\alpha}\right)\cdot x_\alpha^\perp$$
$$= \frac{Q_x^2\gamma}{2|x_\alpha|^2}\left((Q_xBR)_\alpha\cdot x_\alpha^\perp + b_s x_{\alpha\alpha}\cdot x_\alpha^\perp + b_e x_{\alpha\alpha}\cdot x_\alpha^\perp + f_\alpha\cdot x_\alpha^\perp + \frac{\psi}{|x_\alpha|}x_{\alpha\alpha}\cdot x_\alpha^\perp\right)$$
$$= \frac{Q_x^2\gamma}{2|x_\alpha|^2}\left((Q_xBR)_\alpha\cdot x_\alpha^\perp + \left(b_s + \frac{\psi}{|x_\alpha|}\right)x_{\alpha\alpha}\cdot x_\alpha^\perp\right)$$
$$+ \frac{Q_x^2\gamma}{2|x_\alpha|^2}\left(b_e x_{\alpha\alpha}\cdot x_\alpha^\perp + f_\alpha\cdot x_\alpha^\perp\right)$$
$$= \frac{Q_x^2\gamma}{2|x_\alpha|^2}\left((Q_xBR)_\alpha\cdot x_\alpha^\perp + \frac{Q_x^2\gamma}{2|x_\alpha|^2}x_{\alpha\alpha}\cdot x_\alpha^\perp\right) + \text{ errors}$$
$$= \frac{Q_x^2\gamma}{2|x_\alpha|^2}G_x(\alpha) + \text{ errors} = \text{NICE } + \text{ errors}$$

Finally, the most singular terms in $Q_x^2\sigma_x$ are

$$L = Q_x^2BR_t\cdot x_\alpha^\perp + \frac{Q_x^2\psi}{|x_\alpha|}BR_\alpha\cdot x_\alpha^\perp$$

We take 3 derivatives and consider the most dangerous characters:

$$L = M_1 + M_2 + M_3 + \text{ l.o.t}$$
$$M_1 = Q_x^2BR(x,\partial_\alpha^3\gamma_t)\cdot x_\alpha^\perp + \frac{Q_x^2\psi}{|x_\alpha|}BR(x,\partial_\alpha^4\gamma)\cdot x_\alpha^\perp$$
$$M_2 = Q_x^2\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\partial_\alpha^3x_t(\alpha)-\partial_\alpha^3x_t(\alpha-\beta))\cdot x_\alpha(\alpha)}{|x(\alpha)-x(\alpha-\beta)|^2}\gamma(\alpha-\beta)d\beta$$
$$+ \frac{Q_x^2\psi}{|x_\alpha|}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\partial_\alpha^4x(\alpha)-\partial_\alpha^4x(\alpha-\beta))\cdot x_\alpha(\alpha)}{|x(\alpha)-x(\alpha-\beta)|^2}\gamma(\alpha-\beta)d\beta$$
$$M_3 = -\frac{Q_x^2}{\pi}\int_{-\pi}^{\pi}\frac{\Delta_\beta x(\alpha)\cdot x_\alpha(\alpha)}{|\Delta_\beta x(\alpha)|^4}\Delta_\beta x(\alpha)\cdot\Delta_\beta\partial_\alpha^3x_t(\alpha)\gamma(\alpha-\beta)d\beta$$
$$-\frac{\psi Q_x^2}{|x_\alpha|\pi}\int_{-\pi}^{\pi}\frac{\Delta_\beta x(\alpha)\cdot x_\alpha(\alpha)}{|\Delta_\beta x(\alpha)|^4}\Delta_\beta x(\alpha)\cdot\Delta_\beta\partial_\alpha^4x(\alpha)\gamma(\alpha-\beta)d\beta$$

In M_2 we find

$$M_2 = \frac{Q_x^2\gamma}{2|x_\alpha|^2}\Lambda(\partial_\alpha^3x_t\cdot x_\alpha) + \frac{Q_x^2\psi\gamma}{|x_\alpha|^3}\Lambda(\partial_\alpha^4x\cdot x_\alpha) + \text{ l.o.t}$$

For the second term we use the usual trick

$$\partial_\alpha^4 x \cdot x_\alpha = -3\partial_\alpha^3 x \cdot x_{\alpha\alpha}$$

For the first term we remember that

$$\begin{aligned}
&|x_\alpha|^2 = A(t) \Rightarrow x_\alpha \cdot x_{\alpha t} = \frac{1}{2}A'(t) \Rightarrow (x_\alpha \cdot x_{\alpha t})_\alpha = 0 \\
&\Rightarrow x_{\alpha\alpha} \cdot x_{\alpha t} + x_\alpha \cdot x_{\alpha\alpha t} = 0 \Rightarrow x_{\alpha\alpha\alpha} \cdot x_{\alpha t} + 2x_{\alpha\alpha} \cdot x_{\alpha\alpha t} + x_\alpha \cdot x_{\alpha\alpha\alpha t} = 0 \\
&\Rightarrow x_\alpha \cdot x_{\alpha\alpha\alpha t} = -2x_{\alpha\alpha} \cdot x_{\alpha\alpha t} - x_{\alpha\alpha\alpha} \cdot x_{\alpha t}
\end{aligned}$$

This allows us to control M_2. For M_3 we find

$$M_3 = -\frac{Q_x^2\gamma}{|x_\alpha|^2}\Lambda(x_\alpha \cdot \partial_\alpha^3 x_t) - \frac{Q_x^2\psi\gamma}{|x_\alpha|^3}\Lambda(x_\alpha \cdot \partial_\alpha^4 x) + \text{ l.o.t}$$

so it can be estimated as M_2. It remains M_1.

$$M_1 = Q_x^2 BR(x, \partial_\alpha^3\gamma_t) \cdot x_\alpha^\perp + \frac{Q_x^2\psi}{|x_\alpha|}BR(x, \partial_\alpha^4\gamma) \cdot x_\alpha^\perp$$

Using that $\Delta_\beta x^\perp(\alpha) \cdot x_\alpha^\perp(\alpha) = \Delta_\beta x(\alpha) \cdot x_\alpha(\alpha)$ we find

$$M_1 = \frac{Q_x^2}{2}H(\partial_\alpha^3\gamma_t) + \frac{Q_x^2\psi}{2|x_\alpha|}H(\partial_\alpha^4\gamma) + \text{ l.o.t} \tag{5.16}$$

We compute

$$\begin{aligned}
\frac{Q_x^2}{2}H(\partial_\alpha^3\gamma_t) &= H\left(\partial_\alpha^3\left(\frac{Q_x^2\gamma}{2}\right)_t\right) + \text{ NICE} \\
&= H(\partial_\alpha^3(|x_\alpha|\psi)_t) + H(\partial_\alpha^3(|x_\alpha|b_s)_t) + \text{ NICE} \\
&= |x_\alpha|H(\partial_\alpha^3\psi_t) + H(\partial_\alpha^2\partial_t(-(Q_x^2BR)_\alpha \cdot x_\alpha)) + \text{ NICE}
\end{aligned} \tag{5.17}$$

We compute the most singular term in

$$\begin{aligned}
\partial_\alpha^2\partial_t(-(Q_x^2BR)_\alpha \cdot x_\alpha) = &-\frac{Q_x^2}{2\pi}\int_{-\pi}^{\pi}\frac{(\partial_\alpha^3 x_t(\alpha) - \partial_\alpha^3 x_t(\alpha-\beta))^\perp \cdot x_\alpha(\alpha)}{|\Delta_\beta x(\alpha)|^2}\gamma(\alpha-\beta)d\beta \\
&+ \underbrace{\frac{Q_x^2}{\pi}\int_{-\pi}^{\pi}\frac{(\Delta_\beta x(\alpha))^\perp \cdot x_\alpha}{|\Delta_\beta x(\alpha)|^4}\Delta_\beta x(\alpha)\Delta_\beta\partial_\alpha^3 x_t(\alpha)\gamma(\alpha-\beta)d\beta}_{\text{extra cancelation}} \\
&- \underbrace{\frac{Q_x^2}{2\pi}\int_{-\pi}^{\pi}\frac{(\Delta_\beta x(\alpha))^\perp \cdot x_\alpha}{|\Delta_\beta x(\alpha)|^2}\partial_\alpha^3\gamma_t(\alpha-\beta)d\beta}_{\text{extra cancelation}} + \text{l.o.t.} \ + \text{ NICE}
\end{aligned}$$

This shows that

$$\partial_\alpha^2\partial_t(-(Q_x^2BR)_\alpha\cdot x_\alpha)=-\frac{Q_x^2\gamma}{2|x_\alpha|^2}\Lambda(\partial_\alpha^3x_t^\perp\cdot x_\alpha)+\text{ l.o.t. }+\text{ NICE}$$

That gives

$$\partial_\alpha^2\partial_t(-(Q_x^2BR)_\alpha\cdot x_\alpha)=-\Lambda\left(\frac{Q_x^2\gamma}{2|x_\alpha|^2}\partial_\alpha^3x_t^\perp\cdot x_\alpha\right)+\text{ l.o.t. }+\text{ NICE}$$

which implies

$$\begin{aligned}H(\partial_\alpha^2\partial_t(-(Q_x^2BR)_\alpha\cdot x_\alpha))&=\partial_\alpha\left(\frac{Q_x^2\gamma}{2|x_\alpha|^2}\partial_\alpha^3x_t^\perp\cdot x_\alpha\right)+\text{ l.o.t. }+\text{ NICE}\\&=-\frac{Q_x^2\gamma}{2|x_\alpha|^2}\partial_\alpha\left(\partial_\alpha^3x_t\cdot x_\alpha^\perp\right)+\text{ NICE}\end{aligned}$$

Plugging the above formula in (5.17) we find that

$$\begin{aligned}\frac{Q_x^2}{2}H(\partial_\alpha^3\gamma_t)&=|x_\alpha|H(\partial_\alpha^3\psi_t)-\frac{Q_x^2\gamma}{2|x_\alpha|^2}\partial_\alpha\left(\partial_\alpha^3x_t\cdot x_\alpha^\perp\right)+\text{ NICE}\\&=|x_\alpha|H(\partial_\alpha^3\psi_t)-\frac{Q_x^2\gamma}{2|x_\alpha|^2}\partial_\alpha\left(\partial_\alpha^3(Q_x^2BR)\cdot x_\alpha^\perp\right)-\frac{Q_x^2\gamma}{2|x_\alpha|^2}\partial_\alpha\left(b_s\partial_\alpha^4x\cdot x_\alpha^\perp\right)+\text{ l.o.t}\\&\quad+\text{ NICE}+\text{ errors}\end{aligned}$$

As we did before, in $\partial_\alpha(\partial_\alpha^3(Q_x^2BR)\cdot x_\alpha^\perp)$, the most dangerous term is given by $Q_x^2\frac{1}{2}H(\partial_\alpha^4\gamma)$, the tangential terms appear, which implies

$$\partial_\alpha(\partial_\alpha^3(Q_x^2BR)\cdot x_\alpha^\perp)=Q_x^2\frac{1}{2}H(\partial_\alpha^4\gamma)+\text{ NICE}$$

and therefore

$$\frac{Q_x^2}{2}H(\partial_\alpha^3\gamma_t)=|x_\alpha|H(\partial_\alpha^3\psi_t)-\frac{Q_x^2\gamma}{2|x_\alpha|^2}\frac{Q_x^2}{2}H(\partial_\alpha^4\gamma)-\frac{Q_x^2\gamma}{2|x_\alpha|^2}b_s\partial_\alpha\left(\partial_\alpha^4x\cdot x_\alpha^\perp\right)+\text{ NICE}+\text{ errors}$$

We use (5.16) to find

$$\begin{aligned}&\frac{Q_x^2}{2}H(\partial_\alpha^3\gamma_t)+\frac{Q_x^2\psi}{2|x_\alpha|}H(\partial_\alpha^4\gamma)\\&\quad=|x_\alpha|H(\partial_\alpha^3\psi_t)-\frac{Q_x^2}{2}b_sH(\partial_\alpha^4\gamma)-\frac{Q_x^2\gamma}{2|x_\alpha|^2}b_s\partial_\alpha\left(\partial_\alpha^4x\cdot x_\alpha^\perp\right)+\text{ NICE}+\text{ errors}\\&\quad=|x_\alpha|H(\partial_\alpha^3\psi_t)-b_s|x_\alpha|H\left(\partial_\alpha^4\left(\frac{Q_x^2\gamma}{2|x_\alpha|}\right)\right)-\frac{Q_x^2\gamma}{2|x_\alpha|^2}b_s\partial_\alpha\left(\partial_\alpha^4x\cdot x_\alpha^\perp\right)+\text{ NICE}+\text{ errors}\\&\quad=|x_\alpha|H(\partial_\alpha^3\psi_t)-b_s|x_\alpha|H\left(\partial_\alpha^4\psi\right)-b_s|x_\alpha|H(\partial_\alpha^4(b_s|x_\alpha|))-\frac{Q_x^2\gamma}{2|x_\alpha|^2}b_s\partial_\alpha\left(\partial_\alpha^4x\cdot x_\alpha^\perp\right)\\&\quad+\text{ NICE}+\text{ errors}\end{aligned}$$

We will show that

$$-b_s|x_\alpha|H(\partial_\alpha^4(b_s|x_\alpha|)) - \frac{Q_x^2\gamma}{2|x_\alpha|^2}b_s\partial_\alpha\left(\partial_\alpha^4 x\cdot x_\alpha^\perp\right)$$

is NICE and then we are done.

$$\begin{aligned}-b_s|x_\alpha|H(\partial_\alpha^4(b_s|x_\alpha|)) - \frac{Q_x^2\gamma}{2|x_\alpha|^2}b_s\partial_\alpha\left(\partial_\alpha^4 x\cdot x_\alpha^\perp\right) &= -b_sH(\partial_\alpha^4(b_s|x_\alpha|^2)) - \frac{Q_x^2\gamma}{2|x_\alpha|^2}b_s\partial_\alpha\left(\partial_\alpha^4 x\cdot x_\alpha^\perp\right)\\ &= b_sH(\partial_\alpha^3((Q_x^2BR)_\alpha\cdot x_\alpha)) - \frac{Q_x^2\gamma}{2|x_\alpha|^2}b_s\partial_\alpha\left(\partial_\alpha^4 x\cdot x_\alpha^\perp\right)\end{aligned}$$

We repeat the calculation for dealing with the most dangerous terms in

$$\partial_\alpha^3((Q_x^2BR)_\alpha\cdot x_\alpha) = \Lambda\left(\partial_\alpha^4 x^\perp\cdot x_\alpha\frac{\gamma Q_x^2}{2|x_\alpha|^2}\right) + \text{ l.o.t}$$

In the l.o.t we use that $\Delta_\beta x^\perp(\alpha)\cdot x(\alpha)$ gives an extra cancelation. We find that

$$\begin{aligned}&b_sH(\partial_\alpha^3((Q_x^2BR)_\alpha\cdot x_\alpha)) - \frac{Q_x^2\gamma}{2|x_\alpha|^2}b_s\partial_\alpha\left(\partial_\alpha^4 x\cdot x_\alpha^\perp\right)\\ &\qquad= b_sH(\Lambda\left(\partial_\alpha^4 x^\perp\cdot x_\alpha\frac{\gamma Q_x^2}{2|x_\alpha|^2}\right)) - \frac{Q_x^2\gamma}{2|x_\alpha|^2}b_s\partial_\alpha\left(\partial_\alpha^4 x\cdot x_\alpha^\perp\right) + \text{ NICE}\\ &\qquad= -b_s\partial_\alpha\left(\partial_\alpha^4 x^\perp\cdot x_\alpha\frac{\gamma Q_x^2}{2|x_\alpha|^2}\right) - \frac{Q_x^2\gamma}{2|x_\alpha|^2}b_s\partial_\alpha\left(\partial_\alpha^4 x\cdot x_\alpha^\perp\right) + \text{ l.o.t} + \text{ NICE}\end{aligned}$$

Using that $\partial_\alpha^4 x^\perp\cdot x_\alpha = -\partial_\alpha^4 x\cdot x_\alpha^\perp$ we are done.

Chapter 6

Turning waves for the inhomogeneous Muskat problem: a computer-assisted proof

6.1 Introduction

The evolution of a fluid in a porous medium is an interesting problem in fluid mechanics [15, 77]. Darcy, in 1856 tried to formulate the laws of a water flow through vertical homogeneous sand filters (a porous medium). Without taking gravity into account, he postulated the following relation (see Figure 6.1):

$$\mu \frac{Q}{A} = \kappa \frac{P_l - P_r}{L}, \tag{6.1}$$

where

- Q is the total discharge (Vol/t).
- A is the cross section of the pipe.
- K is the length between the measure points.
- μ is the viscosity of the fluid.
- κ is the permeability of the medium.
- P_l and P_r are the pressure at the left and right ends respectively.

For a continuous medium, if we also reflect the effect of gravity, Darcy's law is given by

$$\frac{\mu}{\kappa} v = -\nabla p - (0, g\rho), \tag{6.2}$$

where

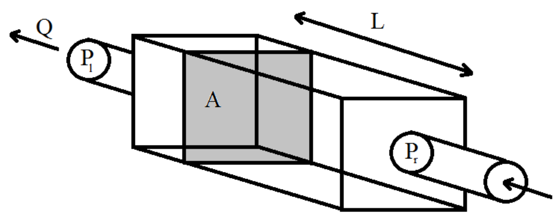


Figure 6.1: Darcy's device for his experiment. Water flows through a porous medium.

- g is the gravitational acceleration constant.
- ρ is the density of the fluid.
- v is the velocity of the fluid.

Darcy's law has been verified many times experimentally and can be derived from Navier-Stokes' equations using homogenization methods [86].

Muskat, in 1937 [74], studied the evolution of ground water and its interaction with oil in a sandy medium by looking at the interface that separated the two fluids. Therefore, this is nowadays known as the Muskat problem.

Darcy's law presents many similarities with the movement of a fluid trapped in a Hele-Shaw cell, which was studied by Hele-Shaw in 1898 [59, 60]. A Hele-Shaw cell consists of two thin parallel vertical plates, situated at a distance b which we will assume to be very small compared to the area of the plates (see Figure 6.2). Starting from the Stokes equations and setting $v = (v_1, v_2, v_3)$ as the velocity of the fluid we get

$$\rho(v \cdot \nabla v) = -\nabla p + \mu \Delta v - (0, 0, g\rho), \quad \mathrm{div}(v) = 0. \tag{6.3}$$

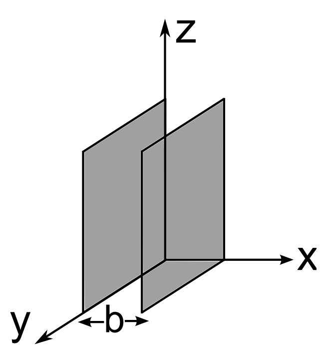


Figure 6.2: A Hele-Shaw cell.

Assuming that b is very small and that the plates are situated orthogonally to the x-axis, the velocity will depend only on y and z. Substituting this into (6.3), we get

$$
\begin{aligned}
0 &= -\partial_x p,\\
\rho(v_2\partial_y v_2 + v_3\partial_z v_2) &= -\partial_y p + \mu\Delta v_2,\\
\rho(v_2\partial_y v_3 + v_3\partial_z v_3) &= -\partial_z p + \mu\Delta v_3 - \rho g.
\end{aligned}
$$

We can also assume that the derivatives of v_2 and v_3 in the y, z directions are small compared to the ones in the x direction. We can therefore approximate the previous system by

$$
\begin{aligned}
\partial_x p &= 0,\\
\partial_y p &= \mu\partial_{xx} v_2,\\
\rho g + \partial_z p &= \mu\partial_{xx} v_3.
\end{aligned}
$$

Since p does not depend on x, and v takes vales 0 at the boundary ($x = 0$ and $x = b$) we can integrate the previous equations to get

$$\mu v_2 = \frac{1}{2}(x^2 - bx)\partial_y p, \tag{6.4}$$

$$\mu v_3 = \frac{1}{2}(x^2 - bx)(\partial_z p + \rho g). \tag{6.5}$$

Finally, averaging over every $x \in [0, b]$ the mean velocity $\overline{v}$ can be written as

$$\frac{12\mu}{b^2}\overline{v} = -\nabla p - (0, g\rho), \tag{6.6}$$

which is analogous to Darcy's law by setting the permeability of the medium κ equal to $\frac{b^2}{12}$. Saffman and Taylor [80] studied the evolution of the interface between water and oil on a Hele-Shaw cell, obtaining the same equations as in the Muskat problem. This is why the equations for the Muskat problem are also known as the two-phase Hele-Shaw equations. We refer the reader to the papers [7], [17], [22], [27], [29], [31], [32] and specially the survey paper [23] for the most important results concerning the unconfined Muskat problem. From now on, we will work in the two dimensional case, although the generalization to the 3D one is immediate.

The Muskat problem has also been studied in what is called the confined regime. In the confined regime the two incompressible fluids can not penetrate into a "top" and a "bottom" layers which we will assume are at height L and $-L$ (see Figure 6.3). Moreover, we will consider that both fluids have the same viscosity but different densities (ρ^1 the upper fluid, ρ^2 the lower one) and we will denote by $z(\alpha, t) = (z_1(\alpha, t), z_2(\alpha, t))$ the interface of the free boundary, which can be either horizontally periodic or flat at infinity. Thus, our system of equations is

$$\begin{cases} \frac{\mu}{\kappa} v &= -\nabla p - \rho(0,g), \\ \nabla \cdot v &= 0, \\ \rho_t + v \cdot \nabla \rho &= 0, \\ v_2(x,-L,t) &= v_2(x,L,t) = 0. \end{cases}$$

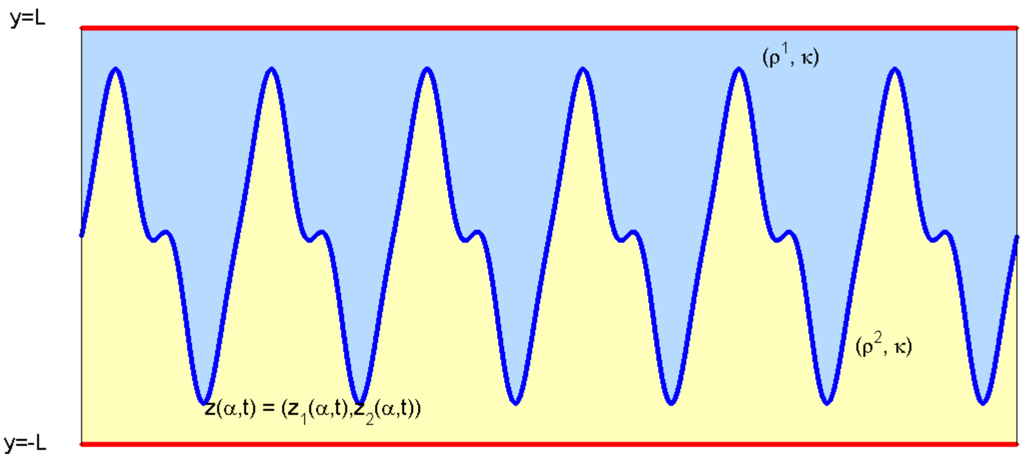


Figure 6.3: The confined Muskat problem. Sketch of the situation.

since we don't take into account the effects of surface tension. Manipulating the system, one can get an evolution equation for $z(\alpha,t)$ (setting $L = \frac{\pi}{2}$ and $\rho^2 - \rho^1 = 4\pi$):

$$\partial_t z(\alpha,t) = \text{P.V.} \int_{\mathbb{R}} \left[\frac{(\partial_\alpha z(\alpha) - \partial_\alpha z(\eta)) \sinh(z_1(\alpha) - z_1(\eta))}{\cosh(z_1(\alpha) - z_1(\eta)) - \cos(z_2(\alpha) - z_2(\eta))} \right. \\ \left. + \frac{(\partial_\alpha z_1(\alpha) - \partial_\alpha z_1(\eta), \partial_\alpha z_2(\alpha) + \partial_\alpha z_2(\eta)) \sinh(z_1(\alpha) - z_1(\eta))}{\cosh(z_1(\alpha) - z_1(\eta)) + \cos(z_2(\alpha) + z_2(\eta))} \right] d\eta, \quad (6.7)$$

or, if the interface is parametrized as a graph $(\alpha, f(\alpha))$,

$$\partial_t f(\alpha,t) = \text{P.V.} \int_{\mathbb{R}} \left[(\partial_\alpha f(\alpha) - \partial_\alpha f(\alpha - \eta)) \frac{\sinh(\eta)}{\cosh(\eta) - \cos(f(\alpha) - f(\alpha - \eta))} \right. \\ \left. + (\partial_\alpha f(\alpha) + \partial_\alpha f(\alpha - \eta) \frac{\sinh(\eta)}{\cosh(\eta) + \cos(f(\alpha) + f(\alpha - \eta))} \right] d\eta. \quad (6.8)$$

For the confined model, local existence, a maximum principle and global existence for a class of initial data were proved in [33] in the case without surface tension. For the case with surface tension, local existence in a certain Hölder space was proved in [40], in which the authors also study bifurcations of the stationary solutions in the unstable case with surface tension. Similar results for the case with three fluids and two interfaces were discussed in [39].

We could also think of a model for the Muskat problem that also incorporates a jump in the permeabilities of the medium. this means that we can write the permeability of the medium as

$$\kappa(x) = \kappa_1 1_{\{x \in \Omega^1\}} + \kappa_2 1_{\{x \in \Omega^2\}},$$

where Ω^1 and Ω^2 respectively denote the space below or above a given boundary. This model has gained importance since it has been used for many applications: for example the description of a geothermal reservoir, oil exploration, soil physics, ground water hydrology, etc. (see [25], [38] and the references therein).

Again, Darcy's law governs the movement of the velocity of the fluids, which also have a jump of densities across an interface. We will again denote by $z(\alpha, t) = (z_1(\alpha, t), z_2(\alpha, t))$ the interface and by $b(\alpha) = (h_1(\alpha), h_2(\alpha))$ the (fixed) boundary at which the permeability jump is placed. Moreover, we will assume that this boundary is given by $h_1(\alpha) = \alpha$, $h_2(\alpha) = -h_2$ for a constant $L > h_2 > 0$ (see Figure 6.4). We will designate by $\mathcal{K} = \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2}$ the adimensional parameter relating the different permeabilities. It is easy to see that by definition $-1 < \mathcal{K} < 1$.

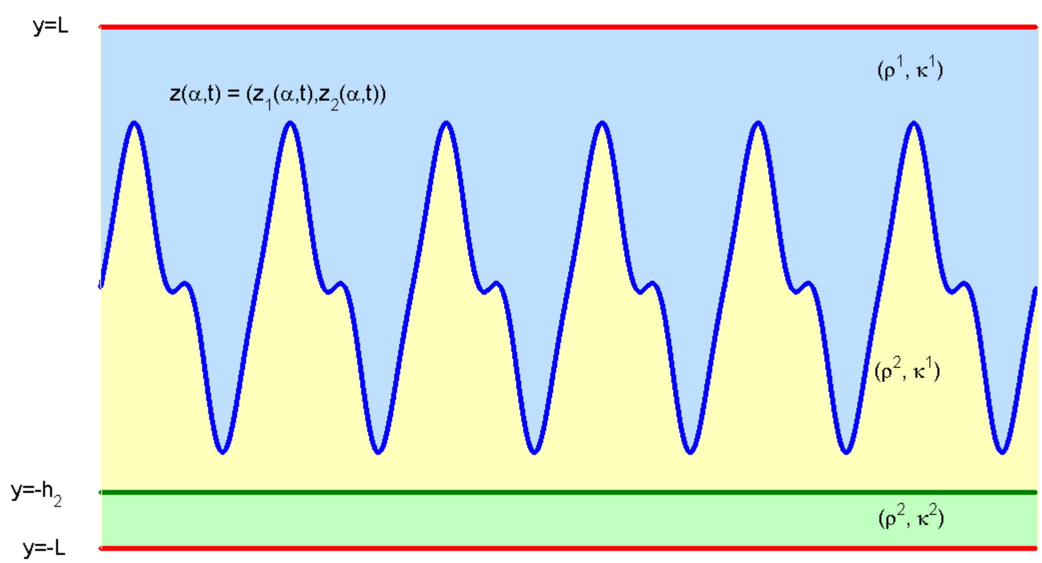


Figure 6.4: The inhomogeneous Muskat problem. Sketch of the situation.

In the next sections we will work in the inhomogeneous, non confined regime. A comparison between the inhomogeneous confined and non confined regimes, as well as a quantitative description of some interfaces that can develop turning singularities in both regimes can be found in [52].

The organization of this chapter is as follows: a precise statement of the theorems is given in the next section, the technical details can be found in Section 6.3 and the codes in Appendix B. In order to perform the rigorous computations we will use the C-XSC library [61].

6.2 Main Results

In this section we will state the theorems that will be proved in the next one. We show that the fact of having a confined medium plays a role in the mechanism for achieving turning singularities. Moreover, we also show that there are cases for which the jump in the permeabilities can lead to either prevent or promote these singularities, and cases in which the heterogeneity of the medium has no impact on whether the wave turns or not.

Theorem 6.2.1 *There exists a family of analytic curves $z(\alpha) = (z_1(\alpha), z_2(\alpha))$, flat at infinity, for which there exists a finite time T such that the confined Muskat problem develops a turning singularity before $t = T$ and the non confined does not.*

Theorem 6.2.2 *(a) There exists a family of analytic curves $z(\alpha) = (z_1(\alpha), z_2(\alpha))$, periodic in the horizontal variable and a finite time T, for which the inhomogeneous, non confined Muskat problem develops a turning singularity at time T independently of the permeability parameter $\mathcal{K}$.*

(b) There exists a family of analytic curves $z(\alpha) = (z_1(\alpha), z_2(\alpha))$, periodic in the horizontal variable, for which the inhomogeneous, non confined Muskat problem develops a turning singularity or not, depending on the value of $\mathcal{K}$. More precisely, there exist a finite time T and constants K_1, $K_2 > 0$ for which for every $-1 < \mathcal{K} < K_1$ the curves have not turned over at time T and for every $K_2 < \mathcal{K} < 1$ they have turned.

Theorem 6.2.3 *(a) There exists a family of analytic curves $z(\alpha) = (z_1(\alpha), z_2(\alpha))$, that are flat at infinity and a finite time T, for which the inhomogeneous, non confined Muskat problem develops a turning singularity at time T independently of the permeability parameter $\mathcal{K}$.*

(b) There exists a family of analytic curves $z(\alpha) = (z_1(\alpha), z_2(\alpha))$, that are flat at infinity, for which the inhomogeneous, non confined Muskat problem develops a turning singularity or not, depending on the value of $\mathcal{K}$. More precisely, there exist a finite time T and constants K_1, $K_2 < 0$ for which for every $-1 < \mathcal{K} < K_1$ the curves have turned over at time T and for every $K_2 < \mathcal{K} < 1$ they have not.

Remark 6.2.4 *We should remark that Theorems 6.2.2 and 6.2.3 are more general than the ones in [16, Theorem 3, Theorem 4] since we are suppressing any smallness assumption in $|\mathcal{K}|$ or largeness in h_2.*

6.3 Technical details concerning the proofs

In this section we will outline the technical details concerning the computer-assisted part of the proof of the theorems stated in the previous section. For the details concerning the analytical part of the theorems, see [53] and [52].

We should remark that throughout the section, each proof will be built upon the previous ones. This is also reflected in the code, in which we always strive for a balance between readability, simplicity, length of the code and execution time: the first proofs are simpler and less optimized but more readable, while as we advance throughout the proofs we develop more complicated algorithms, keeping in mind that the complexity order should be optimal but without trying to do any low-level optimization, in order to provide a more understandable code. Every proof can be checked in a personal desktop computer under 10 seconds, which we believe is a reasonable time.

From now on, suppose that aside from the previous hypotheses we also assume the following additional conditions on the initial datum z:

- $z_i(\alpha)$ are odd functions,
- $\partial_\alpha z_1(0) = 0, \partial_\alpha z_1(\alpha) > 0 \ \forall \alpha \neq 0$, and $\partial_\alpha z_2(0) > 0$,
- $|z_2(\alpha)| \neq h_2(\alpha) \ \forall \ \alpha$.

Proof of Theorem 6.2.1:

In order to get a curve such that for the confined problem a singularity in the form of the interface ceasing to be a graph appears, we are left to validate the following sign for the integrals (we assume we have taken $l = \frac{\pi}{2}$ and $\rho^2 - \rho^1 = 4\pi$):

$$\begin{aligned} 0 > I^A_{neg} = \partial_\alpha z_2(0) \int_0^\infty & \partial_\alpha z_1(\alpha) \sinh(z_1(\alpha)) \sin(z_2(\alpha)) \\ & \times \left(\frac{1}{(\cosh(z_1(\alpha)) - \cos(z_2(\alpha)))^2} + \frac{1}{(\cosh(z_1(\alpha)) + \cos(z_2(\alpha)))^2} \right) d\alpha. \end{aligned} \tag{6.9}$$

In the case of the unconfined problem, the phenomenon of not turning (for a short enough time) is equivalent to prove that

$$0 < I^A_{pos} = \partial_\alpha z_2(0) \int_0^\infty \frac{\partial_\alpha z_1(\alpha) \sin(z_1(\alpha)) \sinh(z_2(\alpha))}{(\cosh(z_2(\alpha)) - \cos(z_1(\alpha)))^2} d\alpha. \tag{6.10}$$

Thus, it suffices to validate conditions (6.9)-(6.10). We rigorously validate them for the following data:

$$z_1(\alpha) = \alpha - \sin(\alpha)e^{-K\alpha^2}, \quad K = 10^{-4}$$

$$z_2(\alpha) = \begin{cases} \dfrac{\sin(3\alpha)}{3} & \text{if } 0 \leq \alpha \leq \dfrac{\pi}{3} \\ -\alpha + \dfrac{\pi}{3} & \text{if } \dfrac{\pi}{3} \leq \alpha \leq \dfrac{\pi}{2} \\ \alpha - \dfrac{2\pi}{3} & \text{if } \dfrac{\pi}{2} \leq \alpha \leq \dfrac{2\pi}{3} \\ 0 & \text{if } \dfrac{2\pi}{3} \leq \alpha, \end{cases}$$

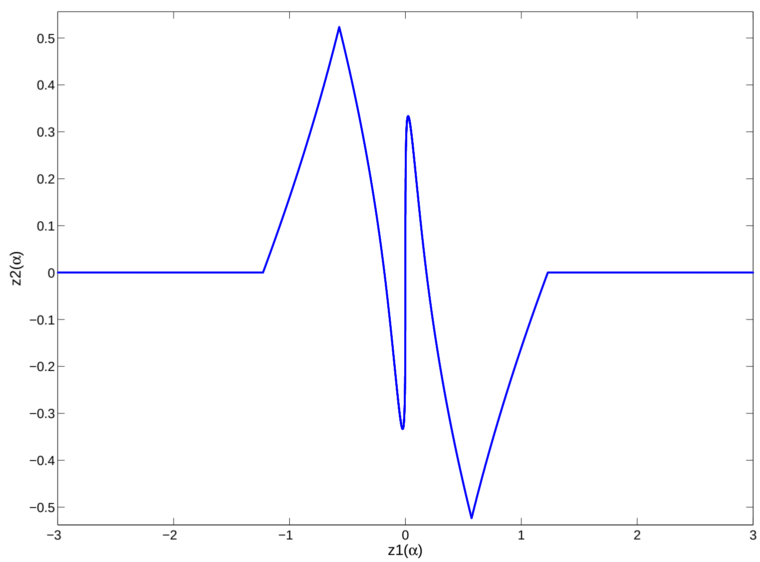


Figure 6.5: Initial Data for Theorem 6.2.1.

where z_2 is extended such that it is an odd function. We split each of I^A_{pos} and I^A_{neg} into three pieces, each corresponding to a different piece of the piecewise defined z_2 in which $z_2(\alpha)$ is not identically 0.

In the second and third pieces, the integrand is analytic and we can apply Simpson's rule on a uniform (equispaced) mesh $\eta_0 < \eta_1 < \ldots < \eta_{N+1}$ for the computation of the integrals:

$$\int_a^b f(\eta)d\eta = \sum_{i=0}^{N} \int_{\eta_i}^{\eta_{i+1}} f(\eta)d\eta = \sum_{i=0}^{N} \frac{(\eta_{i+1} - \eta_i)}{6} \left(f(\eta_i) + f(\eta_{i+1}) + 4f\left(\frac{\eta_i + \eta_{i+1}}{2}\right)\right) - \frac{1}{2880}(\eta_{i+1} - \eta_i)^5 f^4([\eta_i, \eta_{i+1}]).$$

The first integral needs special care since the integrand is of type $\frac{0}{0}$ when α goes to zero. We should remark that the function is integrable since the numerator is $O(\alpha^6)$ and the denominator is $O(\alpha^4)$ when expanded both around $\alpha = 0$ in the two problematic cases, namely I^A_{pos} and the first summand of I^A_{neg}. We further split the integral into two pieces, one ranging from 0 to ε and another from ε to $\frac{\pi}{3}$. In the validation of the theorem, the choice of the constant ε equal to $\frac{1}{128}$ was enough. The integrand of the second piece is analytic and is calculated as before, while for the first piece we expand both the numerator and the denominator and cancel out the extra factors α. In our case this means (for I^A_{pos}):

$$\int_0^\varepsilon \frac{\partial_\alpha z_1(\alpha)\sin(z_1(\alpha))\sinh(z_2(\alpha))}{(\cosh(z_2(\alpha)) - \cos(z_1(\alpha)))^2}d\alpha \equiv \int_0^\varepsilon \frac{N(\alpha)}{D(\alpha)}d\alpha \in \int_0^\varepsilon \frac{\sum_{i=0}^5 a_i\alpha^i + \frac{1}{6!}\partial_\alpha^6 N([0,\varepsilon])\alpha^6}{\sum_{j=0}^3 b_j\alpha^j + \frac{1}{4!}\partial_\alpha^4 D([0,\varepsilon])\alpha^4}d\alpha.$$

Since $a_0, \ldots, a_5, b_0, \ldots, b_3$ are zero, we get

$$\int_0^\varepsilon \frac{N(\alpha)}{D(\alpha)}d\alpha \in \int_0^\varepsilon \frac{4!}{6!}\frac{[0,\varepsilon]^2\partial_\alpha^6 N([0,\varepsilon])}{\partial_\alpha^4 D([0,\varepsilon])}d\alpha \subset \frac{[0,\varepsilon]^3}{120}\frac{\partial_\alpha^6 N([0,\varepsilon])}{\partial_\alpha^4 D([0,\varepsilon])}$$

The code is flexible so that N can be specified by the user of the program. One can see that for small values of N, the intervals in which the value of I^A_{pos}, I^A_{neg} are enclosed are not small enough such that 0 doesn't belong to them, needing further precision. However, for $N = 8192$ the grid is fine enough to check conditions (6.9)-(6.10). In this setting, there is no need for multiprecision and a double representation of 64 bits is enough. We obtain for $N = 8192$ the following results:

$$\begin{aligned} I^A_{pos} &\in [0.021217219225477916555444457, 0.021217380138465771061140 34], \\ I^A_{neg} &\in [-0.013681973445849869228108 10, -0.013681812861882745521735 49]. \end{aligned}$$

The computation took 3.70 seconds on an Intel i5 processor with 4 GB of RAM. To finish the theorem, it is enough to take as initial data a perturbation of z_2 in such a way that it is analytic and fulfills conditions (6.9)-(6.10).

□

Proof of Theorem 6.2.2: In this case, the question whether the interface turns over or not is reduced to find a sign (negative resp. positive) of

$$\partial_\alpha z_2(0)\left(\int_0^\pi \frac{\partial_\alpha z_1(\beta)\sin(z_1(\beta))\sinh(z_2(\beta))}{(\cosh(z_2(\beta)) - \cos(z_1(\beta)))^2}d\beta \right. \tag{6.11}$$

$$\left. + \frac{1}{4\pi}\int_0^\pi \frac{(\omega^B(\beta) + \omega^B(-\beta))(-1 + \cosh(h_2)\cos(\beta))}{(\cosh(h_2) - \cos(\beta))^2}d\beta\right), \tag{6.12}$$

where ω^B is defined by

$$\omega^B(\beta) = \mathcal{K}\int_{-\pi}^\pi \frac{\sin(\beta - z_1(\gamma))\partial_\alpha z_1(\gamma)}{\cosh(h_2 + z_2(\gamma)) - \cos(\beta - z_1(\gamma))}d\gamma \tag{6.13}$$

and we assume $\kappa^1(\rho^2-\rho^1)=2\pi$. Plugging (6.13) into (6.12) we have to compute

$$\begin{aligned}
I^B &\equiv \partial_\alpha z_2(0)\left(\int_0^\pi \frac{\partial_\alpha z_1(\beta)\sin(z_1(\beta))\sinh(z_2(\beta))}{(\cosh(z_2(\beta))-\cos(z_1(\beta)))^2}d\beta\right.\\
&+\frac{\mathcal{K}}{4\pi}\int_0^\pi\int_{-\pi}^\pi \frac{\sin(\beta-z_1(\gamma))\partial_\alpha z_1(\gamma)(-1+\cosh(h_2)\cos(\beta))}{(\cosh(h_2)-\cos(\beta))^2}\\
&\times\left.\left(\frac{1}{\cosh(h_2+z_2(\gamma))-\cos(\beta-z_1(\gamma))}+\frac{1}{\cosh(h_2+z_2(\gamma))-\cos(-\beta-z_1(\gamma))}\right)d\beta d\gamma\right)\\
&\equiv I_1^B+I_2^B.
\end{aligned} \tag{6.14}$$

We remark that the integrand of the 2D integral above is regular (does not even have an indetermination such as the 1D one) since we are assuming that $|z_2(\alpha)|<h_2$. We calculate I_1^B as in the first case. However, the choice of a uniform grid in I_2^B leads to high execution times or low precision. In order to ameliorate the performance of the algorithm, we will integrate in an adaptive way. We will start with the full domain $[0,\pi]\times[-\pi,\pi]$ and in each iteration we will use a 2D Simpson's rule.

$$\begin{aligned}
\int_a^b\int_c^d f(x,y)dxdy &= \frac{(b-a)(d-c)}{36}\left(16f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\\
&+4\left(f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\right)\\
&+\left.\left(f(b,c)+f(b,d)+f(a,c)+f(a,d)\right)\right)\\
&-\frac{(b-a)(d-c)}{2880}\left((b-a)^4\partial_x^4 f([a,b],[c,d])+(d-c)^4\partial_y^4 f([a,b],[c,d])\right).
\end{aligned}$$

If the result meets some tolerance requirements in the form of the result having absolute or relative (with respect to the volume of the integration region) width smaller than two constants (AbsTol and RelTol) we will save the result and add it to the total. Otherwise, we bisect our domain in each of the two directions and call the integrator again with the new 4 subdomains recursively. We also keep track of the number of calls to the integrator in order to prevent infinite loops or stack overflows because of too stringent tolerances, but this was not necessary for the parameters specified below.

In order to prove the theorem we will take the following curves defined for $\alpha\in[-\pi,\pi]$ and extended periodically in the horizontal variable.

$$\begin{aligned}
z_1(\alpha) &= \alpha-\sin(\alpha),\\
z_2(\alpha) &= \begin{cases} \dfrac{\sin(3\alpha)}{3}-\sin(\alpha)\left(e^{-(\alpha+2)^2}+e^{-(\alpha-2)^2}\right) & \text{in case } (a)\\ \dfrac{\sin(2\alpha)}{2}-\dfrac{2}{3}\sin(\alpha)\left(e^{-(\alpha+2)^2}+e^{-(\alpha-2)^2}\right) & \text{in case } (b).\end{cases}
\end{aligned}$$

After running the program with the previous data we get the following results:

(a) Choosing the parameters $N = 8192$, RelTol $= 10^{-5}$, AbsTol $= 10^{-5}$, $\mathcal{K} = 1$, $h_2 = \frac{\pi}{2}$ we got the following results:

$$I_1^B \in [-0.7910700276626275728730775\,0, -0.7910698174361350254457647\,6],$$
$$I_2^B \in [-0.1270343628080658204826392\,0, -0.1269936769973550316770172\,2],$$
$$I^B \in [-0.9181043904706934766224435\,5, -0.9180634944334900016116307\,5]$$

The running time of the code was 6.59 seconds and there were 7305 calls to the recursive integrator. We can see that I_1^B is in absolute value bigger than I_2^B and therefore the total result is negative independent of the value of $\mathcal{K}$, which will result in the curve turning over.

(b) Choosing the parameters $N = 8192$, RelTol $= 10^{-5}$, AbsTol $= 10^{-5}$, $\mathcal{K} = 1$, $h_2 = \frac{\pi}{2}$ we got the following results:

$$I_1^B \in [0.1243125192075989482454190\,2, 0.1243127223872871889298608\,1],$$
$$I_2^B \in [-0.1414549310884918031927526\,3, -0.1414142293221318236184913\,5],$$
$$I^B \in [-0.0171424118808928549473336\,1, -0.0171015069348446346886305\,4]$$

The running time of the code was 5.32 seconds and there were 5677 calls to the recursive integrator. We can see that I_1^B is in absolute value smaller than I_2^B. This means that there exist some values K_1, K_2 such that for all $-1 < \mathcal{K} < K_1$ the curve does not turn over and for all $K_2 < \mathcal{K} < 1$ the curve turns over. In this case, the different permeabilities of the medium help in the formation of singularities.

□

Proof of Theorem 6.2.3:

The turning or not (for a short enough time) for the flat at infinity case can be shown to be equivalent to finding a sign of

$$I^C \equiv \partial_\alpha z_2(0)\left(4\text{P.V.}\int_0^\infty \frac{\partial_\alpha z_1(\beta) z_1(\beta) z_2(\beta)}{((z_1(\beta))^2 + (z_2(\beta))^2)^2} d\beta - \frac{1}{2\pi}\text{P.V.}\int_0^\infty \frac{(\omega^C(\beta) + \omega^C(-\beta))\beta^2}{(\beta^2 + h_2^2)^2} d\beta\right), \tag{6.15}$$

where ω^C is defined by

$$\omega^C(\beta) = 2\mathcal{K}\text{P.V.}\int_{-\infty}^{\infty} \frac{(h_2 + z_2(\gamma))\partial_\alpha z_2(\gamma)}{(h_2 + z_2(\gamma))^2 + (\beta - z_1(\gamma))^2} d\gamma \tag{6.16}$$

and we assume $\kappa^1(\rho^2 - \rho^1) = 2\pi$. Plugging (6.16) into (6.15) we have to compute

$$\begin{aligned} I^C \quad & \equiv \partial_\alpha z_2(0)\Bigg(4\text{P.V.}\int_0^\infty \frac{\partial_\alpha z_1(\beta) z_1(\beta) z_2(\beta)}{((z_1(\beta))^2 + (z_2(\beta))^2)^2} d\beta \\ & -\frac{\mathcal{K}}{\pi}\int_0^\infty \int_{-\infty}^\infty \frac{(h_2 + z_2(\gamma))\partial_\alpha z_2(\gamma)\beta^2}{(\beta^2 + h_2^2)^2} \\ & \times\left(\frac{1}{(h_2 + z_2(\gamma))^2 + (\beta - z_1(\gamma))^2} + \frac{1}{(h_2 + z_2(\gamma))^2 + (-\beta - z_1(\gamma))^2}\right) d\beta d\gamma\Bigg) \\ & \equiv I_1^C + I_2^C. \end{aligned} \tag{6.17}$$

Again, we compute I_1^C as in Theorem 6.2.1. It is important to notice that we are now integrating I_2^C in an unbounded region. Even in the case that z_2 has compact support and the integral in γ is different than zero in a compact set, the integral in β cannot be reduced to integrate in a bounded region. Therefore, we split I_2^C into a bounded part and an unbounded one. We now explain how to deal with the latter since the former is computed as in the previous Theorem.

We want to bound

$$T \equiv -\frac{\mathcal{K}}{\pi}\partial_\alpha z_2(0)\int_M^\infty\int_{-\infty}^\infty \frac{(h_2+z_2(\gamma))\partial_\alpha z_2(\gamma)\beta^2}{(\beta^2+h_2^2)^2}\left(\frac{1}{(h_2+z_2(\gamma))^2+(\beta-z_1(\gamma))^2}\right.$$
$$\left.+\frac{1}{(h_2+z_2(\gamma))^2+(-\beta-z_1(\gamma))^2}\right)d\beta d\gamma$$

and we will take the following curves.

$$z_1(\alpha)=\alpha-\sin(\alpha)e^{-K\alpha^2},\quad K=10^{-2},$$
$$z_2(\alpha)=\begin{cases}\left(\dfrac{\sin(3\alpha)}{3}-\sin(\alpha)\left(e^{-(\alpha+2)^2}+e^{-(\alpha-2)^2}\right)\right)1_{\{|\alpha|\leq\pi\}} & \text{in case } (a)\\ \left(\dfrac{\sin(1.88\alpha)}{1.88}-\sin(\alpha)\left(e^{-(\alpha+2)^2}+e^{-(\alpha-2)^2}\right)\right)1_{\{|\alpha|\leq\pi\}} & \text{in case } (b).\end{cases}$$

We will provide bounds for T in this way:

$$|T|\leq\frac{|\mathcal{K}|}{\pi}|\partial_\alpha z_2(0)|\int_M^\infty\frac{\beta^2}{(\beta^2+h_2^2)^2}d\beta\int_{-\pi}^{\pi}\left(\frac{|h_2+z_2(\gamma)||\partial_\alpha z_2(\gamma)|}{(h_2+z_2(\gamma))^2+(\beta-z_1(\gamma))^2}\right.$$
$$\left.+\frac{|h_2+z_2(\gamma)||\partial_\alpha z_2(\gamma)|}{(h_2+z_2(\gamma))^2+(-\beta-z_1(\gamma))^2}\right)d\gamma \tag{6.18}$$

and let

$$G(\beta)\equiv\frac{|\mathcal{K}|}{\pi}\int_{-\pi}^{\pi}\left(\frac{|h_2+z_2(\gamma)||\partial_\alpha z_2(\gamma)|}{(h_2+z_2(\gamma))^2+(\beta-z_1(\gamma))^2}+\frac{|h_2+z_2(\gamma)||\partial_\alpha z_2(\gamma)|}{(h_2+z_2(\gamma))^2+(-\beta-z_1(\gamma))^2}\right)d\gamma.$$

It is easy to check that $G(\beta)$ is monotone in β for β larger than $\|z_1\|_{L^\infty(-\pi,\pi)}$. Indeed,

$$G(\beta)\leq G(M),\qquad \text{if we take } M=14\pi,$$

which is our choice of M for the computer verification. Plugging this relation into (6.18) we obtain

$$|T|\leq|\partial_\alpha z_2(0)|G(M)\int_M^\infty\frac{\beta^2}{(\beta^2+h_2^2)^2}d\beta \tag{6.19}$$
$$=|\partial_\alpha z_2(0)|G(M)\left(\frac{\pi}{4h_2}-\frac{1}{h_2}\arctan\left(\frac{M}{h_2}\right)+\frac{M}{2(h_2^2+M^2)}\right) \tag{6.20}$$

Thus, we are left to compute rigorous bounds for G. Let us denote by

$$IG(\beta,\gamma)=\frac{|\mathcal{K}|}{\pi}\left(\frac{|h_2+z_2(\gamma)||\partial_\alpha z_2(\gamma)|}{(h_2+z_2(\gamma))^2+(\beta-z_1(\gamma))^2}+\frac{|h_2+z_2(\gamma)||\partial_\alpha z_2(\gamma)|}{(h_2+z_2(\gamma))^2+(-\beta-z_1(\gamma))^2}\right) \tag{6.21}$$

the integrand of G. That means

$$G(\beta)=\int_{-\pi}^{\pi} IG(\beta,\gamma)d\gamma.$$

We perform the following integration scheme:

$$\int_{\gamma_i}^{\gamma_{i+1}} IG(\beta,\gamma)d\gamma=\begin{cases} IG(\beta,[\gamma_i,\gamma_{i+1}])(\gamma_{i+1}-\gamma_i) & \text{if } 0\in IG(\beta,[\gamma_i,\gamma_{i+1}]) \\ \frac{(\gamma_{i+1}-\gamma_i)}{6}\left(IG(\beta,\gamma_i)+IG(\beta,\gamma_{i+1})+4IG\left(\beta,\frac{\gamma_i+\gamma_{i+1}}{2}\right)\right) & \\ -\frac{1}{2880}(\gamma_{i+1}-\gamma_i)^5\partial_\gamma^4 IG(\beta,[\gamma_i,\gamma_{i+1}]) & \text{otherwise}\end{cases}$$

in which we apply a Simpson rule for the case where the integrand is smooth, otherwise we take the full interval that results in evaluating the integrand in the whole integration interval.

Therefore, adding all the contributions

$$G(M)=\sum_{i=1}^{N_2}\int_{\gamma_i}^{\gamma_{i+1}} IG(M,\gamma)d\gamma,$$

we get the desired bound on T. The variable N_2 is user-specified in our program.

After running the program with the initial data presented before we obtained the following results:

(a) Choosing the parameters $N = 8192$, $N_2 = 256$, RelTol = AbsTol = 10^{-5}, $\mathcal{K} = -1$, $h_2 = \frac{\pi}{2}$ we get

$$\begin{aligned} I_1^C &\in [-0.7456401336460000139894077\,7, -0.74563989994303225827820824], \\ T &\in [-0.000026676998685696427018\,06, 0.00002667699868569642701806], \\ I_2^C - T &\in [0.02034657516565453738710544, 0.02068402037491086017939602], \\ I^C &\in [-0.72532023547903123894542433, -0.72492920256943560453066766] \end{aligned}$$

The running time of the code was 7.56 seconds and there were 9205 calls to the recursive integrator. We can see that I_1^C dominates I_2^C, hence there will be turning for any $-1 < \mathcal{K} < 1$. In this case, the heterogeneity of the medium can not prevent the formation of singularities for any value of the permeabilities.

(b) Choosing the parameters $N = 8192$, $N_2 = 256$, RelTol = AbsTol = 10^{-5}, $\mathcal{K} = 1$, $h_2 = \frac{\pi}{2}$ we get

$$
\begin{aligned}
I_1^C &\in [-0.00059107053222070204349243, -0.00059083812284237949459531],\\
T &\in [-0.00002520339771374320749073, 0.00002520339771374320749073],\\
I_2^C - T &\in [0.00839958726488038848190242, 0.00871016631351071227151728],\\
I^C &\in [0.00778331333494594254651666, 0.00814453158838207742775684]
\end{aligned}
$$

The running time of the code was 7.89 seconds and there were 9421 calls to the recursive integrator. We can see that I_1^C does not dominate I_2^C, hence there will exist K_1, K_2 such that there is turning for all $-1 < \mathcal{K} < K_1$ and no turning for all $K_2 < \mathcal{K} < 1$ for a short enough time. In this case, the heterogeneity of the medium can prevent the formation of singularities for some value of the permeabilities.

$\square$

Appendix A

Splash singularity for water waves: Codes

This appendix contains the code for the simulations of the water waves problem in the tilde domain. The codes have been implemented in Matlab following the scheme developed by Beale, Hou and Lowengrub [13] adapted to the new equations (i.e. incorporating the impact of the function Q). We will present the different files from outside to inside.

A.1 waterwaves_potato.m

Listing A.1: waterwaves_potato.m

```
% Interface for the water waves problem
clear all
close all

format long e;

example = 115;
g = 1; % Gravity
tf = 0.0070;
% Discretization in space
N = 2048;
h = 2*pi/N;
alpha = [-pi+2*pi/N:h:pi]';
rho = zeros(N,1);
q = zeros(N,1);
dphi_ini = zeros(N,1);
phi_ini = zeros(N,1);

if example == 115 % Drop
   load 'drop_trig_2048.mat';
   z_ini = my_sqrt(tan(drop/2));
   z_ini = z_ini.';
```

```
for jj=1:N
    rho(jj) = exp(-10*(2*abs(jj-N/2)/N)^25);
end

Vdhz = dh(z_ini,rho,alpha);
dpsi_ini = zeros(N,1);
CC(1) = 3;
CC(2) = -3.4;
CC(3) = 1;
CC(4) = 0.2;
dpsi_ini = CC(1)*cos(alpha) + CC(2)*cos(2*alpha) + CC(3)*cos(3*alpha)...
          + CC(4)*cos(4*alpha);
A = zeros(N,N);

for ii=1:N
    for jj=1:N
        if (ii ~= jj)
            A(ii,jj) = real(Vdhz(ii)/(2*pi*i)/(z_ini(ii)-z_ini(jj))*h);
        end
   end
end

g_ini = (-A-1/2*eye(N))\dpsi_ini;

for ii=1:N
    for jj=1:N
        if (ii ~= jj)
            phi_ini(ii) = phi_ini(ii) + 1/(2*pi)*g_ini(jj)...
                         *log(abs(z_ini(ii)-z_ini(jj)))*h;
        end
    end
end
dphi_ini = dh(phi_ini,rho,alpha);
dphi_ini = real(dphi_ini);

B = 0*A;

for ii=1:N
   for jj=1:N
         if (mod(ii-jj,2) ~= 0)
              B(ii,jj) = real(Vdhz(ii)/(2*pi*i)/(z_ini(ii)-z_ini(jj))*2*h);
         end
   end
end

gamma_ini = (B+1/2*eye(N))\dphi_ini;

ti = 0;
```

```
    dt = 0.00000002;
    [tvec,sol_ab,gamma,zt] = adams_bashforth_potato_iterative(ti,tf,dt,...
                                    [z_ini; phi_ini],alpha,h,rho,g,gamma_ini);
end
```

A.2 my_sqrt.m

Listing A.2: my_sqrt.m

```
function out = my_sqrt(z)
  ang = my_angle(z);
  modulus = abs(z);
  out = sqrt(modulus).*exp(ang/2*i);
end
```

A.3 my_angle.m

Listing A.3: my_angle.m

```
function ang = my_angle(z)
ang = angle(z);
ang(find(ang < 0)) = ang(find(ang < 0)) + 2*pi; % Positive semiaxis R^+
% We work by continuity, assuming only a difference in the phase of \pm 2*pi
for ii=2:length(z)
    if (abs(ang(ii) +2*pi - ang(ii-1)) < abs(ang(ii) - ang(ii-1)))
        ang(ii) = ang(ii) + 2*pi;
    elseif (abs(ang(ii) -2*pi - ang(ii-1)) < abs(ang(ii) - ang(ii-1)))
        ang(ii) = ang(ii) - 2*pi;
    else
        ang(ii) = ang(ii);
    end
end
```

A.4 adams_bashforth_potato_iterative.m

Listing A.4: adams_bashforth_potato_iterative.m

```
function [tvec,out,gamma,deriv] = adams_bashforth_potato_iterative(ti,T,dt,y_ini,...
                                                 alpha,h,rho,g,gamma_0)
% Fourth order Adams-Bashforth method. For the first 3 values of the
% function, a 4th order Runge-Kutta is used.
tvec = [ti:dt:T]; % We assume T-ti is a multiple of dt.
Tsteps = length(tvec);
NSamples = 1000;
```

```
NBlocks = ceil(Tsteps/NSamples);
currblock = 1;
N = length(y_ini);
deriv = zeros(N,NSamples);
gamma = zeros(N/2,NSamples);
out = zeros(N,NSamples);
N = N/2;
backupfile = sprintf('water_waves_bhl_AB_2H_Candidate_ti_%g_tf_%g_N_%g_step_%g',
                     ti,T,N,dt);
out(:,1) = y_ini;
gamma(:,1) = gamma_0;

% Runge-Kutta Fourth Order

deriv(:,1) = fode_ww_bhl_potato_iterative(tvec(1),out(:,1),alpha,h,rho,g,gamma(:,1));
k1 = deriv(:,1);
k2 = fode_ww_bhl_potato_iterative(tvec(1)+dt/2,out(:,1)+1/2*dt*k1,alpha,...
                                  h,rho,g,gamma(:,1));
k3 = fode_ww_bhl_potato_iterative(tvec(1)+dt/2,out(:,1)+1/2*dt*k2,alpha,...
                                  h,rho,g,gamma(:,1));
k4 = fode_ww_bhl_potato_iterative(tvec(1)+dt,out(:,1)+dt*k3,alpha,h,rho,g,gamma(:,1));
out(:,2) = out(:,1) + 1/6*dt*(k1+2*k2+2*k3+k4);

[deriv(:,2),gamma(:,2)] = fode_ww_bhl_potato_iterative_with_gamma(tvec(2),out(:,2),...
                                                        alpha,h,rho,g,gamma(:,1));
k1 = deriv(:,2);
k2 = fode_ww_bhl_potato_iterative(tvec(2)+dt/2,out(:,2)+1/2*dt*k1,alpha,...
                                  h,rho,g,gamma(:,2));
k3 = fode_ww_bhl_potato_iterative(tvec(2)+dt/2,out(:,2)+1/2*dt*k2,alpha,...
                                  h,rho,g,gamma(:,2));
k4 = fode_ww_bhl_potato_iterative(tvec(2)+dt,out(:,2)+dt*k3,alpha,...
                                  h,rho,g,gamma(:,2));
out(:,3) = out(:,2) + 1/6*dt*(k1+2*k2+2*k3+k4);

[deriv(:,3),gamma(:,3)] = fode_ww_bhl_potato_iterative_with_gamma(tvec(3),out(:,3),...
                                                        alpha,h,rho,g,gamma(:,2));
k1 = deriv(:,3);
k2 = fode_ww_bhl_potato_iterative(tvec(3)+dt/2,out(:,3)+1/2*dt*k1,...
                                                        alpha,h,rho,g,gamma(:,3));
k3 = fode_ww_bhl_potato_iterative(tvec(3)+dt/2,out(:,3)+1/2*dt*k2,...
                                                        alpha,h,rho,g,gamma(:,3));
k4 = fode_ww_bhl_potato_iterative(tvec(3)+dt,out(:,3)+dt*k3,alpha,h,rho,g,gamma(:,3));
out(:,4) = out(:,3) + 1/6*dt*(k1+2*k2+2*k3+k4);

[deriv(:,4),gamma(:,4)] = fode_ww_bhl_potato_iterative_with_gamma(tvec(4),out(:,4),...
                                                        alpha,h,rho,g,gamma(:,3));

fprintf(1,'End of RK-4\n');
%Rest
```

```
for i=5:Tsteps
    x = mod(i,NSamples);
    if (x == 0)
        out(:,NSamples) = out(:,NSamples-1) + dt*(55/24*deriv(:,NSamples-1)...
                          - 59/24*deriv(:,NSamples-2) + 37/24*deriv(:,NSamples-3)...
                          - 3/8*deriv(:,NSamples-4));
        [deriv(:,NSamples),gamma(:,NSamples)] =
               fode_ww_bhl_potato_iterative_with_gamma(tvec(i),out(:,NSamples),...
                                                 alpha,h,rho,g,gamma(:,NSamples-1));
        chunk_backupfile = strcat(backupfile,sprintf('_chunk_%g_of_%g.mat',...
                                                        currblock,NBlocks));
        save(chunk_backupfile,'tvec','out','alpha','gamma','N','deriv');
        currblock = currblock+1;
    elseif (x == 1)
        out(:,1) = out(:,NSamples) + dt*(55/24*deriv(:,NSamples)
                 - 59/24*deriv(:,NSamples-1) + 37/24*deriv(:,NSamples-2)
                 - 3/8*deriv(:,NSamples-3));
        [deriv(:,1),gamma(:,1)] = fode_ww_bhl_potato_iterative_with_gamma(tvec(i),...
                                    out(:,1),alpha,h,rho,g,gamma(:,NSamples-1));
    elseif (x == 2)
        out(:,2) = out(:,1) + dt*(55/24*deriv(:,1) - 59/24*deriv(:,NSamples)...
                 + 37/24*deriv(:,NSamples-1) - 3/8*deriv(:,NSamples-2));
        [deriv(:,x),gamma(:,x)] = fode_ww_bhl_potato_iterative_with_gamma(tvec(i),...
                                    out(:,x),alpha,h,rho,g,gamma(:,x-1));
    elseif (x == 3)
        out(:,3) = out(:,2) + dt*(55/24*deriv(:,2) - 59/24*deriv(:,1)...
                 + 37/24*deriv(:,NSamples) - 3/8*deriv(:,NSamples-1));
        [deriv(:,x),gamma(:,x)] = fode_ww_bhl_potato_iterative_with_gamma(tvec(i),...
                                    out(:,x),alpha,h,rho,g,gamma(:,x-1));
    elseif (x == 4)
        out(:,4) = out(:,3) + dt*(55/24*deriv(:,3) - 59/24*deriv(:,2)...
                 + 37/24*deriv(:,1) - 3/8*deriv(:,NSamples));
        [deriv(:,x),gamma(:,x)] = fode_ww_bhl_potato_iterative_with_gamma(tvec(i),...
                                    out(:,x),alpha,h,rho,g,gamma(:,x-1));
    else
        out(:,x) = out(:,x-1) + dt*(55/24*deriv(:,x-1) - 59/24*deriv(:,x-2)...
                 + 37/24*deriv(:,x-3) - 3/8*deriv(:,x-4));
        [deriv(:,x),gamma(:,x)] = fode_ww_bhl_potato_iterative_with_gamma(tvec(i),...
                                    out(:,x),alpha,h,rho,g,gamma(:,x-1));
    end
end
chunk_backupfile = strcat(backupfile,sprintf('_chunk_%g_of_%g.mat',...
                                                        currblock,NBlocks));
save(chunk_backupfile,'tvec','out','alpha','gamma','N','deriv');
currblock = currblock+1;
end
```

A.5 fode_ww_bhl_potato_iterative.m

Listing A.5: fode_ww_bhl_potato_iterative.m

```
function out = fode_ww_bhl_potato_iterative(t,y,alpha,h,rho,g,gamma_ini)
fprintf('Testing t = %e\n',t);
N = length(y)/2;
z = y(1:N); phi = y(N+1:end);
Vdhz = dh(z,rho,alpha);
Vzp = zp(z,rho,alpha);
Vdhphi = dh(phi,rho,alpha);
gamma1 = compute_gamma_iterative(Vzp,Vdhz,Vdhphi,h,gamma_ini);
[dz, dphi] = dzphi_potato(z,Vzp,Vdhz,gamma1,h,g);
out = [-dz;-dphi]; % We go backwards in time
end
```

A.6 fode_ww_bhl_potato_iterative_with_gamma.m

Listing A.6: fode_ww_bhl_potato_iterative_with_gamma.m

```
function [out,gamma1] = fode_ww_bhl_potato_iterative_with_gamma(t,y,alpha,h,rho,g,...
                                                                 gamma_ini)
fprintf('Testing t = %e\n',t);
N = length(y)/2;
z = y(1:N); phi = y(N+1:end);
Vdhz = dh(z,rho,alpha);
Vzp = zp(z,rho,alpha);
Vdhphi = dh(phi,rho,alpha);
gamma1 = compute_gamma_iterative(Vzp,Vdhz,Vdhphi,h,gamma_ini);
[dz, dphi] = dzphi_potato(z,Vzp,Vdhz,gamma1,h,g);
out = [-dz;-dphi]; % We go backwards in time
end
```

A.7 dh.m

Listing A.7: dh.m

```
function out = dh(f,rho,alpha)
    % Computes a derivative of the function f, f is 2*pi-periodic
    N = length(f);
    out = iftrans(ftrans(f,alpha).*rho.*[-N/2+1:1:N/2]'*i,alpha);
end
```

A.8 zp.m

Listing A.8: zp.m

```
function out = zp(z,rho,alpha)
    % Smooths out a function z, z is 2*pi-periodic
    out = iftrans(ftrans(z,alpha).*rho,alpha);

    % In case z - alpha is periodic (nontilde variables)
    % s = z - alpha;
    % sp = iftrans(ftrans(s,alpha).*rho,alpha);
    % out = sp + alpha;
end
```

A.9 ftrans.m

Listing A.9: ftrans.m

```
function fout = ftrans(u,alpha)
    % Computes the FFT of u in the same way as in the Beale-Hou-Lowengrub paper
    N = length(u);
    fout = 0*u;
    % Equispaced points
    fout = fft(u.*exp(2*pi*i/N*(N/2 - 1)*[1:N]')).*1/N...
          .*exp(2*pi*i/N*(N/2 - 1)*[1:N]')*exp(2*pi*i/N)*exp(-2*pi*i/N*N*N/4);
end
```

A.10 iftrans.m

Listing A.10: iftrans.m

```
function fout = iftrans(u,alpha)
    % Computes the IFFT of u in the same way as in the Beale-Hou-Lowengrub paper
    N = length(u);
    fout = 0*u;
    % Equispaced points
    fout = ifft(u.*exp(-2*pi*i/N*(N/2 - 1)*[1:N]')).*N...
          .*exp(-2*pi*i/N*(N/2 - 1)*[1:N]')*exp(-2*pi*i/N)*exp(2*pi*i/N*N*N/4);
end
```

A.11 compute_gamma_iterative.m

Listing A.11: compute_gamma_iterative.m

```
function gamma = compute_gamma_iterative(zp,dhz,dhphi,h,gamma_ini)
    % Inverts the operator acting on gamma in an iterative way, looking for a fixed
    % point
    N = length(zp);
    A = zeros(N,N);
    for ii=1:2:N
        A(ii,2:2:N) = real(dhz(ii)./(2*pi*i)./(zp(ii)-zp(2:2:N))*2*h);
        A(ii+1,1:2:N) = real(dhz(ii+1)./(2*pi*i)./(zp(ii+1)-zp(1:2:N))*2*h);
    end
    dhphi = real(dhphi);
    gamma_new = 2*dhphi - 2*A*gamma_ini;
    iterations = 1;
    while (norm(gamma_new - gamma_ini) > 1e-8 && iterations < 1000)
        gamma_ini = gamma_new;
        gamma_new = 2*dhphi - 2*A*gamma_ini;
        iterations = iterations + 1;
    end
    fprintf(1,'Iterations = %d\n',iterations);
    gamma = gamma_new;
end
```

A.12 dzphi_potato.m

Listing A.12: dzphi_potato.m

```
function [dz,dphi] = dzphi_potato(z,zp,dhz,gamma,h,g)
% Computes the derivative in time both of phi and z.
% It uses the previously calculated gamma via the inversion of the
% Birkhoff-Rott operator
    N = length(z);
    dz = 0*z; dphi = dz;
    for ii=1:2:N
        dz(ii) = dz(ii) + 1/(2*pi*i)*sum(gamma(2:2:N)./(zp(ii)-zp(2:2:N))*2*h);
        dz(ii) = dz(ii) + gamma(ii)/(2*dhz(ii));
        dz(ii+1) = dz(ii+1) + 1/(2*pi*i)*sum(gamma(1:2:N)./(zp(ii+1)-zp(1:2:N))*2*h);
        dz(ii+1) = dz(ii+1) + gamma(ii+1)/(2*dhz(ii+1));
    end
    dz = conj(dz);
    for ii=1:N
        iz(ii) = inverse_potato(z(ii));
    end
    iz = iz.';
    for ii=1:N
        dphi(ii) = 1/2*abs(dz(ii))^2*abs((z(ii)^4+1)./(4*z(ii))).^2-g*imag(iz(ii));
    end
    dz = dz.*abs((z.^4+1)./(4.*z)).^2;
end
```

A.13 inverse_potato.m

Listing A.13: inverse_potato.m

```
function z = inverse_potato(y)
% Computes the inverse by P
z = 2*atan(y.*y);
end
```

Appendix B

Turning waves for the inhomogeneous Muskat problem: Codes

This appendix contains the code for the rigorous proofs concerning the turning (or not turning) phenomenon for the Muskat problem. The codes have been programmed using the C-XSC library [61]. In the first section we show extra functions added to the library while in the other ones we exhibit the code concerning the theorems proved in Chapter 6.

B.1 Additional functions for C-XSC

In this section we show the methods incorporated to the itaylor and dim2taylor classes in order to differentiate an object and get another object of the same type. Sofar, only the coefficients were available and no outer manipulation of the data is advised since it is a private attribute of the class. We also implement a method that truncates a taylor series (it returns an itaylor/dim2taylor object of a smaller order) so that one can combine it with the differentiation operator in order to work with the itaylor/dim2taylor objects (we recall that operations between itaylor/dim2taylor objects are only allowed if they have the same order). Finally, we propose a modification of the constructor of the dim2taylor class to initialize it with zeros.

B.1.1 Code added to 'itaylor.hpp'

Listing B.1: Headers of the new functions in the itaylor class: differentiation and truncation of the Taylor series

```
friend itaylor diff(const itaylor& x, int order);
friend itaylor trunc(const itaylor& x, int order);
```

B.1.2 Code added to 'itaylor.cpp'

Listing B.2: Additional functionality to the itaylor class: differentiation and truncation of the Taylor series

```
//Differentiation of an itaylor object returned as an itaylor
itaylor diff(const itaylor& x, int order){
  if (order > get_order(x)){
    return itaylor(0,0.0);
  }
  itaylor res(get_order(x)-order);
  for (int i = 0; i <= get_order(x)-order; i++) {
    res.tayl[i]= x.tayl[i+order];
    for(int j = 1; j<=order; j++)
      res.tayl[i]*=(i+j);
  }
  return res;
}


//-------------------------------------------------------------------------
// Truncates an itaylor object up to order 'order'.
// Useful to combine it with derivation of the itaylor
// objects and further operations
itaylor trunc(const itaylor& x, int order){
  if (order < 0){
    std::cerr << "Error in trunc_itaylor, new_order < 0" << std::endl;
    exit(1);
  }
  else if (order > get_order(x)){
    std::cerr << "Error in trunc_itaylor, new_order > order(x)" << std::endl;
    exit(1);
  }
  else {
    itaylor res(order);
    for (int i = 0; i <= order; i++){
      res.tayl[i] = x.tayl[i];
    }
    return res;
  }
}
```

B.1.3 Code added to 'dim2taylor.hpp'

Listing B.3: Headers of the new functions in the dim2taylor class: differentiation and truncation of the Taylor series

```
  friend dim2taylor diff(const dim2taylor&, int, int); // added, javi: 2013-03
  friend dim2taylor trunc(const dim2taylor&, int); // added, javi: 2013-03
```

B.1.4 Code added to 'dim2taylor.cpp'

Listing B.4: Additional functionality to the dim2taylor class: differentiation and truncation of the Taylor series

```
// Differentiation function //added, javi: 2013-03
// Differentiates diff_x times in the first variable
// Differentiates diff_y times in the second variable
dim2taylor diff(const dim2taylor& d, int diff_x, int diff_y){
  if (diff_x + diff_y > d.order()){
    return dim2taylor(0);
  }
  int new_order = d.order() - diff_x - diff_y;
  dim2taylor res(new_order);
  for (int i = 0; i <= new_order; i++){
    for (int j = 0; j <= new_order - i; j++){
      res.dat[i][j] = d.dat[i+diff_x][j+diff_y];
      for (int k_x = 1; k_x <= diff_x; k_x++){
        res.dat[i][j] *= (i + k_x);
      }
      for (int k_y = 1; k_y <= diff_y; k_y++){
        res.dat[i][j] *= (j + k_y);
      }
    }
  }
  return res;
}

//------------------------------------------------------------------------

// Truncates a dim2taylor object up to order 'order'. // added, javi: 08.03.13
// Useful to combine it with differentiation of the dim2taylor
// objects and further operations
dim2taylor trunc(const dim2taylor& d, int order){
  if (order < 0){
        std::cerr << "Error in trunc_itaylor, new_order < 0" << std::endl;
        exit(1);
  }
  else if (order > d.order()){
        std::cerr << "Error in trunc_itaylor, new_order > order(previous)"
              << std::endl;
        exit(1);
  }
  else {
    dim2taylor res(order);
    for (int i = 0; i <= order; i++){
      for (int j = 0; j <= order - i; j++){
        res.dat[i][j] = d.dat[i][j];
      }
    }
```

```
    return res;
  }
}
```

Listing B.5: Constructor for the dim2taylor class

```
dim2taylor::dim2taylor(int order)
{
 p=order;
 dat=new ivector[p+1];
 for(int i=0; i<=p; i++) Resize(dat[i], 0, p-i);

 for (int i=0; i <= p; i++){
  for (int j=0; j <= p-i; j++){
    dat[i][j] = (interval)0.0;
  }
 }
}
```

B.2 Theorem 6.2.1

Listing B.6: Proof of Theorem 6.2.1

```
// Proof of Theorem 6.2.1. Case Flat at infinity.

// We validate the sign of two integrals, a positive and a negative one,
// concerning a function of the velocity in the non-confined regime and the
// confined respectively.
// The integrals are split into three parts since some of the functions are
// piecewise defined, see below. Moreover, a singularity occurs at alpha = 0,
// in which the integral becomes singular in the sense that the integrand
// is of type 0/0 in the limit, although continuous.
// To overcome this difficulty, we integrate in a small neighborhood of zero,
// doing Taylor series both in the numerator and the denominator.
// In order to speed up some calculations, some functions are duplicated
// depending on the input/output data types (either interval or itaylor).
// Every integral (except the singular part) is computed using Simpson's
// rule on a uniform mesh of N subintervals, where N is specified by the user.
//
// All calculations are done using rigorous arithmetics using the C-XSC library.

// Usage: ./Theorem_1 N
// N = Number of intervals in which we discretize the integrals

// Output:
//
// RESULTS: (N = 8192)
// Positive integral = [ 0.021217219225477916555444457, 0.021217380138465771061140341]
```

```
// Negative integral = [-0.013681973445849869228108l0,-0.01368181286188274552173549]

// DIAMETERS
// 0.00000016091298785450569575
// 0.00000016058396712370637260


#include "interval.hpp"
#include "itaylor.hpp"
#include <iostream>
#include <cassert>

// Maximum number of points of the quadrature
#define MAXN 5000000
// Maximum order of the Taylor expansion for the Newton-Cotes quadratures.
#define ORDER_TAYLOR 4
// Maximum order of the Taylor expansion of the singularity at 0.
#define ORDER_TAYLOR_SINGULARITY 6

using namespace cxsc;
using namespace std;
using namespace taylor;

//////////////////////////////////////////////////////////
// BEGIN DEFINITION OF THE CURVES
//////////////////////////////////////////////////////////

// Parametrization of the curve: itaylor data type
itaylor z1(const itaylor& alpha, const interval& K){
  return alpha - sin(alpha)*exp(-K*sqr(alpha));
}

// We define each piece of the piecewise defined function by the numbers 0,1 or 2.
itaylor z2(const itaylor& x, int part){
  if (part == 0){
    return sin(3.0*x)/(interval)3.0;
  }
  else if (part == 1){
    return -x + Pid3_interval;
  }
  else if (part == 2){
    return x - 2.0*Pid3_interval;
  }
  // We should never arrive here: there are only 3 pieces!
  cerr << "Incorrect integration part" << endl;
  assert(false);
}

// Parametrization of the curve: interval data type
```

```
interval z1(const interval& alpha, const interval& K){
  return alpha - sin(alpha)*exp(-K*sqr(alpha));
}

interval z2(const interval& x, int part){
  if (part == 0){
    return sin(3.0*x)/(interval)3.0;
  }
  else if (part == 1){
    return -x + Pid3_interval;
  }
  else if (part == 2){
    return x - 2.0*Pid3_interval;
  }
  // We should never arrive here
  cerr << "Incorrect integration part" << endl;
  assert(false);
}

//////////////////////////////////////////////////////////////
// END DEFINITION OF THE CURVES
//////////////////////////////////////////////////////////////

//////////////////////////////////////////////////////////////
// BEGIN DEFINITION OF THE INTEGRANDS
//////////////////////////////////////////////////////////////

// Integrand of the positive integral: interval data type
interval integrand_pos(const interval& alpha, const interval& K, int part_z2){
  itaylor x(1,alpha);
  itaylor Z1 = z1(x,K);
  interval Z2 = z2(alpha,part_z2);
  // We do this to avoid recomputing Z1
  itaylor DZ1 = diff(Z1, 1);
  Z1 = trunc(Z1,0);
  // Now, Z1 and DZ1 are 0-order itaylor objects
  itaylor integrand = (interval)4.0*DZ1*Z1*Z2/(sqr(sqr(Z1)+sqr(Z2)));
  return get_j_derive(integrand,0);
}


// Integrand of the negative integral: interval data type
interval integrand_neg(const interval& alpha, const interval& K, int part_z2){
  itaylor x(1,alpha);
  itaylor Z1 = z1(x,K);
  interval Z2 = z2(alpha,part_z2);
  // We do this to avoid recomputing Z1
  itaylor DZ1 = diff(Z1, 1);
  Z1 = trunc(Z1,0);
```

```
  // Now, Z1 and DZ1 are 0-order itaylor objects
  itaylor integrand = DZ1*sinh(Z1)*sin(Z2)*((interval)1.0/sqr(cosh(Z1) - cos(Z2))
                                        +(interval)1.0/sqr(cosh(Z1) + cos(Z2)));
  return get_j_derive(integrand,0);
}


// Integrand of the positive integral: itaylor data type
itaylor integrand_pos(const interval& x, const interval& K, int part_z2, int order){
  itaylor x_it = itaylor(order+1,x);
  itaylor Z1 = z1(x_it,K);
  itaylor DZ1 = diff(Z1, 1);
  Z1 = trunc(Z1,order);
  x_it = trunc(x_it,order);
  itaylor Z2 = z2(x_it,part_z2);
  // Now, Z1 and DZ1 are 'order'-order itaylor objects
  return (interval)4.0*DZ1*Z1*Z2/(sqr(sqr(Z1)+sqr(Z2)));
}

// Integrand of the negative integral: itaylor data type
itaylor integrand_neg(const interval& x, const interval& K, int part_z2, int order){
  itaylor x_it = itaylor(order+1,x);
  itaylor Z1 = z1(x_it,K);
  itaylor DZ1 = diff(Z1, 1);
  Z1 = trunc(Z1,order);
  x_it = trunc(x_it,order);
  itaylor Z2 = z2(x_it,part_z2);
  // Now, Z1 and DZ1 are 'order'-order itaylor objects
  return DZ1*sinh(Z1)*sin(Z2)*((interval)1.0/sqr(cosh(Z1) - cos(Z2))
                             +(interval)1.0/sqr(cosh(Z1) + cos(Z2)));
}




// Singular part of the positive integral
interval singularity_pos(const interval& alpha, const interval& K){
  itaylor x(ORDER_TAYLOR_SINGULARITY+1,alpha);
  itaylor Z1 = z1(x,K); itaylor DZ1 = diff(Z1,1);
  Z1 = trunc(Z1,ORDER_TAYLOR_SINGULARITY);
  x = trunc(x,ORDER_TAYLOR_SINGULARITY);
  itaylor Z2 = z2(x,0);
  // taylor on the numerator
  itaylor num = sinh(Z1)*sin(Z2)*DZ1;
  // taylor on the singular part of the denominator
  itaylor den1 = sqr(cosh(Z1)-cosh(Z2));
  // Den is O(alpha^4), Num is O(alpha^6) when expanded around alpha = 0;
  // For the other summand, we expand up to alpha^6 in num, up to alpha^0 in den.
  itaylor den2 = sqr(cosh(Z1)+cos(Z2));
```

```
  return get_j_coef(num,6)/get_j_coef(den1,4)*sqr(alpha)
       + get_j_coef(num,6)*power(alpha,6)/get_j_coef(den2,0);
}

// Singular part of the negative integral
interval singularity_neg(const interval& alpha, const interval& K){
  itaylor x(ORDER_TAYLOR_SINGULARITY+1,alpha);
  itaylor Z1 = z1(x,K); itaylor DZ1 = diff(Z1,1);
  Z1 = trunc(Z1,ORDER_TAYLOR_SINGULARITY);
  x = trunc(x,ORDER_TAYLOR_SINGULARITY);
  itaylor Z2 = z2(x,0);
  // taylor on the numerator
  itaylor num = (interval)4.0*Z1*Z2*DZ1;
  // taylor on the denominator
  itaylor den = sqr(sqr(Z1)+sqr(Z2));
  // Den is O(alpha^4), Num is O(alpha^6) when expanded around alpha = 0;
  return get_j_coef(num,6)/get_j_coef(den,4)*sqr(alpha);
}

//////////////////////////////////////////////////////////////
// END DEFINITION OF THE INTEGRANDS
//////////////////////////////////////////////////////////////

//////////////////////////////////////////////////////////////
// BEGIN DEFINITION OF THE INTEGRATION METHODS
//////////////////////////////////////////////////////////////

// Simpson's method for the positive integral
interval Simpson_integrand_pos(const interval& left_abscissa,
                               const interval& right_abscissa,
                               const interval& K, int part_z2){
  interval mid_interval = (left_abscissa+right_abscissa)/(interval)2.0;
  interval alpha = left_abscissa|right_abscissa;
  interval dx = right_abscissa - left_abscissa;
  interval left = integrand_pos(left_abscissa,K,part_z2);
  interval mid = integrand_pos(mid_interval,K,part_z2);
  interval right = integrand_pos(right_abscissa,K,part_z2);
  itaylor error = integrand_pos(alpha,K,part_z2,ORDER_TAYLOR);
  return (left+4.0*mid+right-sqr(sqr(dx))/(interval)20.0
              *get_j_coef(error,4))/(interval)6.0;
}

// Simpson's method for the negative integral
interval Simpson_integrand_neg(const interval& left_abscissa,
                               const interval& right_abscissa,
                               const interval& K, int part_z2){
  interval mid_interval = (left_abscissa+right_abscissa)/(interval)2.0;
  interval alpha = left_abscissa|right_abscissa;
  interval dx = right_abscissa - left_abscissa;
```

```
  interval left = integrand_neg(left_abscissa,K,part_z2);
  interval mid = integrand_neg(mid_interval,K,part_z2);
  interval right = integrand_neg(right_abscissa,K,part_z2);
  itaylor error = integrand_neg(alpha,K,part_z2,ORDER_TAYLOR);
  return (left+4.0*mid+right-sqr(sqr(dx))/(interval)20.0
              *get_j_coef(error,4))/(interval)6.0;
}

//////////////////////////////////////////////////////////////
// END DEFINITION OF THE INTEGRATION METHODS
//////////////////////////////////////////////////////////////


//////////////////////////////////////////////////////////////
// BEGIN DEFINITION OF THE MESH GENERATION METHODS
//////////////////////////////////////////////////////////////

// We keep the mesh for the three different integrands we have to compute depending
// on z2 (recall that z2 is defined as a piecewise function) stored in an interval
// matrix of 3 x N intervals (N subintervals each integral)

// For the first integral (the one going from 0 to pi/3) we strip out the domain
// in which the integrand becomes "singular" (in the sense of being 0/0).
interval mesh[3][MAXN];

// Builds the mesh for the integral between Left and pi/3. Left accounts for a
// small domain in which the integrand becomes a 0/0 type. We deal with that part
// in function 'singularity_pos'.
void build_mesh1(int N, const interval& Left){
  interval Full; Full = Pid3_interval; Full-= Left;
  for (int i=0;i<=N;i++){
    mesh[0][i] = (i)*Full;
    mesh[0][i] /= N;
    mesh[0][i] += Left;
  }
}

// Builds the mesh for the integral between pi/3 and pi/2.
void build_mesh2(int N){
  interval Full; Full = Pid2_interval; Full-= Pid3_interval;
  for (int i=0;i<=N;i++){
    mesh[1][i] = (i)*Full;
    mesh[1][i] /= N;
    mesh[1][i] += Pid3_interval;
  }
}

// Builds the mesh for the integral between pi/2 and 2*pi/3.
void build_mesh3(int N){
```

```
  interval Full; Full = 2.0*Pid3_interval; Full-= Pid2_interval;
  for (int i=0;i<=N;i++){
    mesh[2][i] = (i)*Full;
    mesh[2][i] /= N;
    mesh[2][i] += Pid2_interval;
  }
}

////////////////////////////////////////////////////////////
// END DEFINITION OF THE MESH GENERATION METHODS
////////////////////////////////////////////////////////////

int main(int argc, char* argv[]){
  cout << SetPrecision(29,26);
  if (argc != 2) {
    cout << "Usage: " << endl;
    cout << argv[0] << " N" << endl;
    cout << "N = Number of intervals in which we discretize the integrals" << endl;
    exit(0);
  }
  int N = atoi(argv[1]);
  // Neighbourhood of the singularity set to 1/128 which is representable exactly.
  interval Left; "[0.0078125,0.0078125]" >> Left;
  interval K; "[0.0001,0.0001]" >> K;
  build_mesh1(N,Left); build_mesh2(N); build_mesh3(N);
  interval respos, resneg; respos = 0.0; resneg = 0.0;

  // Computation of the non-singular part of the integral
  for (int mesh_index = 0; mesh_index < 3; mesh_index++){
    for (int i = 0; i < N; i++){
      interval dx; dx = mesh[mesh_index][i+1] - mesh[mesh_index][i];
      interval Ipos; interval Ineg;
      Ipos = Simpson_integrand_pos(mesh[mesh_index][i],mesh[mesh_index][i+1],
                                   K,mesh_index);
      Ineg = Simpson_integrand_neg(mesh[mesh_index][i],mesh[mesh_index][i+1],
                                   K,mesh_index);
      respos+=Ipos*dx;
      resneg+=Ineg*dx;
    }
  }

  // Computation of the singularity around zero
  interval singpos; singpos = 0.0;
  interval singneg; singneg = 0.0;
  singpos = singularity_pos(Left|0.0,K)*Left;
  singneg = singularity_neg(Left|0.0,K)*Left;

  // Output
  cout << "RESULTS" << endl;
```

```
  cout << "Positive integral = " << respos + singpos << endl;
  cout << "Negative integral = " << resneg + singneg << endl;
  cout << endl << "DIAMETERS" << endl;
  cout << diam(respos+singpos) << endl;
  cout << diam(resneg+singneg) << endl;
  return 0;
}
```

B.3 Theorem 6.2.2

Listing B.7: Proof of Theorem 6.2.2

```
// Proof of Theorem 6.2.2(a/b). Periodic Case.
//
// We use this program to validate two scenarios. In the first one, the
// contribution of the first term is negative, and the contribution of the
// second one is positive (for Kappa = -1), but with value smaller than the
// absolute value of the first one. This shows that regardless of the value
// of Kappa, the curve will turn over.
//
// In the second scenario, the contribution of the first integral cannot
// dominate the contribution of the second one. Hence, there exist values
// of Kappa for which the total sum is negative (and therefore the curve
// turns over) and others for which the total sum is positive (and the curve
// does not turn over).
//
// The first integral is a one-dimensional integral and is split into two
// parts, one corresponding to a singularity (of type 0/0, although continuous
// in the limit) and another corresponding to a smooth integrand.
// This integral (except the singular part) is computed using Simpson's
// rule on a uniform mesh of N subintervals, where N is specified by the user.
//
// The second integral is a two-dimensional integral with a smooth integrand.
// The integration is done in an adaptive way, using a 2D Simpson's rule and
// stopping the integration when the relative and absolute width of the results
// meet some tolerance criteria, otherwise splitting into four regions, cutting
// in each dimension in half. We also split if we have a too big integration
// region and incur into division by zero. The number of recursive calls
// (i.e the number of boxes in which we integrate) is recorded.
//
// We would like to remark that, unlike as in the flat at infinity case, there
// are neither tails nor their estimations in the 2D integral.
//
// In order to speed up some calculations, some functions are duplicated
// depending on the input/output data types (either interval, itaylor or dim2taylor).
// For the sake of clarity, we discarded the use of templates by duplicating code.
//
// All calculations are done using rigorous arithmetics using the C-XSC library.
```

```
// Usage: ./2D_v4_periodic N
// N = Number of intervals in which we discretize the first (1D) integral

// Output (Case a)
// RESULTS (N = 8192, RelTol = 1e-5, AbsTol = 1e-5, Kappa = 1)

// First = [-0.79107002766262757287307750,-0.79106981743613502544576476]
// Second = [-0.12703436280806582048263920,-0.12699367699735503167701722]
// Total = [-0.91810439047069347662244355,-0.91806349443349000161163075]

// DIAMETERS

// 0.00000021022649254742731273
// 0.00004068581071078880562197
// 0.00004089603720347501081278
// Number of recursive calls = 7305

// Output (Case b)
// RESULTS (N = 8192, RelTol = 1e-5, AbsTol = 1e-5, Kappa = 1)

// First = [ 0.12431251920759894824541902, 0.12431272238728718892986081]
// Second = [-0.14145493108849180319275263,-0.14141422932213182361849135]
// Total = [-0.01714241188089285494733361,-0.01710150693484463468863054]

// DIAMETERS

// 0.00000020317968824068444178
// 0.00004070176635997957426127
// 0.00004090494604822025870305
// Number of recursive calls = 5677

#include "interval.hpp"
#include "itaylor.hpp"
#include "dim2taylor.hpp"

#include <iostream>

#define MAXN 500000
using namespace cxsc;
using namespace std;
using namespace taylor;

// Maximum order of the Taylor expansion for the Newton-Cotes quadratures.
#define ORDER_TAYLOR 4
// Maximum order of the Taylor expansion of the singularity at 0.
#define ORDER_TAYLOR_SINGULARITY 6

// Mesh for the first integral
```

```
interval mesh1[MAXN];

// Tolerances for the adaptive integration
const double abs_tol = 1e-5;
const double rel_tol = 1e-5;

// Number of recursive calls in the adaptive integration
int RecCalls = 0;

////////////////////////////////////////////////////////////
// BEGIN DEFINITION OF THE CURVES
////////////////////////////////////////////////////////////

// Parametrization of the curve: itaylor data type
itaylor z1(const itaylor& x){
  return x - sin(x);
}

// Comment or uncomment lines 2/4 depending on the version of the Theorem (a/b)
// one wants to validate.
itaylor z2(const itaylor& x){
  // Case turning independent of Kappa
  // return sin((interval)3.0*x)/(interval)3.0
                  - sin(x)*(exp(-sqr(x+(interval)2.0))
                  +exp(-sqr(x-(interval)2.0)));
  // Case turning dependent of Kappa
  return sin(2.0*x)/(interval)2.0 - (interval)2.0/(interval)3.0*sin(x)
                  *(exp(-sqr(x+(interval)2.0))+exp(-sqr(x-(interval)2.0)));
}

// Parametrization of the curve: interval data type coming from ivector
// This is useful to get fast 0-th order representations of z1 while
// working on the second (2D) integral
interval z1(const ivector& v){
  return v[2] - sin(v[2]);
}

// Comment or uncomment lines 2/4 depending on the version of the Theorem (a/b)
// one wants to validate.
interval z2(const ivector& v){
  // Case turning independent of Kappa
  // return sin((interval)3.0*v[2])/(interval)3.0
                  - sin(v[2])*(exp(-sqr(v[2]+(interval)2.0))
                  +exp(-sqr(v[2]-(interval)2.0)));
  // Case turning dependent of Kappa
  return sin((interval)2.0*v[2])/(interval)2.0
                  - (interval)2.0/(interval)3.0*sin(v[2])
                  *(exp(-sqr(v[2]+(interval)2.0))
                  +exp(-sqr(v[2]-(interval)2.0)));
```

```
}

// Parametrization of the curve: interval data type coming from interval
interval z1(const interval& v){
  return v - sin(v);
}

// Comment or uncomment lines 2/4 depending on the version of the Theorem (a/b)
// one wants to validate.
interval z2(const interval& v){
  // Case turning independent of Kappa
  // return sin((interval)3.0*v)/(interval)3.0
                  - sin(v)*(exp(-sqr(v+(interval)2.0))
                  +exp(-sqr(v-(interval)2.0)));
  // Case turning dependent of Kappa
  return sin((interval)2.0*v)/(interval)2.0 - (interval)2.0/(interval)3.0*sin(v)
                  *(exp(-sqr(v+(interval)2.0))+exp(-sqr(v-(interval)2.0)));
}

// Parametrization of the curve: dim2taylor data type
dim2taylor z1(const dim2taylor_vector& v){
  return v[2] - sin(v[2]);
}

// Comment or uncomment lines 2/4 depending on the version of the Theorem (a/b)
// one wants to validate.
dim2taylor z2(const dim2taylor_vector& v){
  // Case turning independent of Kappa
  // return sin((interval)3.0*v[2])/(interval)3.0
                  - sin(v[2])*(exp(-sqr(v[2]+(interval)2.0))
                  +exp(-sqr(v[2]-(interval)2.0)));
  // Case turning dependent of Kappa
  return sin(2.0*v[2])/(interval)2.0 - (interval)2.0/(interval)3.0*sin(v[2])
                  *(exp(-sqr(v[2]+2))+exp(-sqr(v[2]-2)));
}


//////////////////////////////////////////////////////////
// END DEFINITION OF THE CURVES
//////////////////////////////////////////////////////////

//////////////////////////////////////////////////////////
// BEGIN DEFINITION OF THE INTEGRANDS
//////////////////////////////////////////////////////////

// Singular part of the first integral
interval singularity_p1(const interval& alpha, int& flag){
  itaylor x(ORDER_TAYLOR_SINGULARITY+1,alpha);
  itaylor Z1 = z1(x); itaylor DZ1 = diff(Z1,1);
```

```
  Z1 = trunc(Z1,ORDER_TAYLOR_SINGULARITY);
  x = trunc(x,ORDER_TAYLOR_SINGULARITY);
  itaylor Z2 = z2(x);
  // taylor on the numerator
  itaylor num = sin(Z1)*sinh(Z2)*DZ1;
  // taylor on the denominator
  itaylor den = sqr(cosh(Z2)-cos(Z1));
  // Den is O(alpha^4), Num is O(alpha^6) when expanded around alpha = 0;
  if (0 <= get_j_coef(den,4)){
    // Division by zero
    flag = 1;
    return interval(0.0);
  }
  return get_j_coef(num,6)/get_j_coef(den,4)*sqr(alpha);
}


// Integrand of the first integral: itaylor data type
itaylor integrand_p1(const interval& x, int order, int &flag){
  itaylor x_it = itaylor(order+1,x);
  itaylor Z1 = z1(x_it);
  itaylor DZ1 = diff(Z1, 1);
  Z1 = trunc(Z1,order);
  x_it = trunc(x_it,order);
  itaylor Z2 = z2(x_it);
  // Now, Z1 and DZ1 are 'order'-order itaylor objects
  itaylor den = (sqr(cosh(Z2)-cos(Z1)));
  if (0 <= get_j_coef(den,0)){
    // Division by zero
    flag = 1;
    return itaylor(0);
  }
  return DZ1*sin(Z1)*sinh(Z2)/den;
}


// Integrand of the first integral: interval data type
interval integrand_p1(const interval& x, int& flag){
  itaylor x_it = itaylor(1,x);
  itaylor Z1 = z1(x_it);
  itaylor DZ1 = diff(Z1, 1);
  Z1 = trunc(Z1,0);
  interval Z2 = z2(x);
  // Now, Z1 and DZ1 are 'order'-order itaylor objects
  itaylor den = (sqr(cosh(Z2)-cos(Z1)));
  if (0.0 <= get_j_coef(den,0)){
    // Division by zero
    flag = 1;
    return interval(0.0);
```

```
  }
  itaylor integrand = DZ1*sin(Z1)*sinh(Z2)/den;
  return get_j_derive(integrand,0);
}

// Integrand of the second integral: dim2taylor data type
dim2taylor integrand_p2(ivector& iv, const interval& h2,
                        const interval& Kappa, int order, int& flag){
  dim2taylor_vector dv; dv = init_var(order+1,iv);
  dim2taylor_vector dv_low; dv_low = init_var(order,iv);
  dim2taylor Z1 = z1(dv);
  dim2taylor DZ1 = diff(Z1,0,1);
  Z1 = trunc(Z1,order);
  dim2taylor Z2 = z2(dv_low);
  dim2taylor den1 = sqr(cosh(h2)-cos(dv_low[1]));
  dim2taylor den2 = cosh(h2+Z2)-cos(dv_low[1]-Z1);
  dim2taylor den3 = cosh(h2+Z2)-cos(dv_low[1]+Z1);
  if (0.0 <= den1[0][0] || 0.0 <= den2[0][0] || 0.0 <= den3[0][0]){
    // Division by zero
    flag = 1;
    return dim2taylor(0);
  }
  return Kappa/((interval)2.0*Pi2_interval)*(cos(dv_low[1])*cosh(h2)
       - (interval)1.0)/den1*(DZ1)
       *(sin(dv_low[1]-Z1)/den2 - sin(dv_low[1]+Z1)/den3);
}

// Integrand of the second integral: interval data type
interval integrand_p2(const ivector& iv, const interval& h2,
                        const interval& Kappa, int& flag){
  itaylor x_it = itaylor(1,iv[2]);
  itaylor Z1 = z1(x_it);
  interval DZ1 = get_j_derive(Z1,1);
  Z1 = trunc(Z1,0);
  interval Z2 = z2(iv[2]);
  // Now, Z2 and DZ2 are 0-order itaylor objects
  interval den1 = sqr(cosh(h2)-cos(iv[1]));
  itaylor den2 = cosh(h2+Z2)-cos(iv[1]-Z1);
  itaylor den3 = cosh(h2+Z2)-cos(iv[1]+Z1);
  if (0.0 <= den1 || 0.0 <= get_j_coef(den2,0) || 0.0 <= get_j_coef(den3,0)){
    // Division by zero
    flag = 1;
    return interval(0.0);
  }
  itaylor integrand = Kappa/((interval)2.0*Pi2_interval)
        *(cos(iv[1])*cosh(h2)-(interval)1.0)/den1*(DZ1)
        *(sin(iv[1]-Z1)/den2+sin(-iv[1]-Z1)/den3);
  return get_j_derive(integrand,0);
}
```

```
//////////////////////////////////////////////////////////
// END DEFINITION OF THE INTEGRANDS
//////////////////////////////////////////////////////////


//////////////////////////////////////////////////////////
// BEGIN DEFINITION OF THE INTEGRATION METHODS
//////////////////////////////////////////////////////////

// Simpson's method for the first integral
interval Simpson_integrand_p1(const interval& left_abscissa,
                      const interval& right_abscissa, int& flag){
  interval mid_interval = (left_abscissa+right_abscissa)/(interval)2.0;
  interval alpha = left_abscissa|right_abscissa;
  interval dx = right_abscissa - left_abscissa;
  interval left = integrand_p1(left_abscissa,flag);
  interval mid = integrand_p1(mid_interval,flag);
  interval right = integrand_p1(right_abscissa,flag);
  itaylor error = integrand_p1(alpha,ORDER_TAYLOR,flag);
  return (left+(interval)4.0*mid+right-sqr(sqr(dx))/(interval)20.0
                      *get_j_coef(error,4))/(interval)6.0;
}


// Simpson's two-dimensional method for the second integral.
// We assume the integrand is written with the variables ordered as F(beta,gamma)
interval Simpson_integrand_p2(const interval& beta_left, const interval& beta_right,
                           const interval& gamma_down, const interval& gamma_up,
                           const interval& h2, const interval& Kappa, int& flag){
  // Simpson's rule with 9 points
  // Left Up
  ivector left_up_i(2);
  left_up_i[1] = beta_left; left_up_i[2] = gamma_up;
  interval left_up = integrand_p2(left_up_i,h2,Kappa,flag);


  // Center Up
  ivector center_up_i(2);
  center_up_i[1] = (beta_left+beta_right)/(interval)2.0; center_up_i[2] = gamma_up;
  interval center_up = integrand_p2(center_up_i,h2,Kappa,flag);


  // Right Up
  ivector right_up_i(2);
  right_up_i[1] = beta_right; right_up_i[2] = gamma_up;
  interval right_up = integrand_p2(right_up_i,h2,Kappa,flag);
```

```
// Left Center
ivector left_center_i(2);
left_center_i[1] = beta_left; left_center_i[2] = (gamma_up+gamma_down)/(interval)2.0;
interval left_center = integrand_p2(left_center_i,h2,Kappa,flag);


// Center Center
ivector center_center_i(2);
center_center_i[1] = (beta_left+beta_right)/(interval)2.0;
center_center_i[2] = (gamma_up+gamma_down)/(interval)2.0;
interval center_center = integrand_p2(center_center_i,h2,Kappa,flag);


// Right Center
ivector right_center_i(2);
right_center_i[1] = beta_right;
right_center_i[2] = (gamma_up+gamma_down)/(interval)2.0;
interval right_center = integrand_p2(right_center_i,h2,Kappa,flag);


// Left Down
ivector left_down_i(2);
left_down_i[1] = beta_left; left_down_i[2] = gamma_down;
interval left_down = integrand_p2(left_down_i,h2,Kappa,flag);


// Center Down
ivector center_down_i(2);
center_down_i[1] = (beta_left+beta_right)/(interval)2.0;
center_down_i[2] = gamma_down;
interval center_down = integrand_p2(center_down_i,h2,Kappa,flag);


// Right Down
ivector right_down_i(2);
right_down_i[1] = beta_right; right_down_i[2] = gamma_down;
interval right_down = integrand_p2(right_down_i,h2,Kappa,flag);


// Error
ivector error_i(2);
error_i[1] = beta_left|beta_right; error_i[2] = gamma_up|gamma_down;
dim2taylor error = integrand_p2(error_i,h2,Kappa,ORDER_TAYLOR,flag);

if (flag == 1){
  return interval(0.0);
}

interval dx = beta_right - beta_left;
```

```
  interval dy = gamma_up - gamma_down;

  interval error_4x = error[4][0];
  interval error_4y = error[0][4];


  return ((interval)16*center_center
         +(interval)4*(center_up+left_center+right_center+center_down)
         +left_up+right_up+left_down+right_down)/(interval)36.0
         -(error_4x*sqr(sqr(dx))+error_4y*sqr(sqr(dy)))/(interval)120.0;
}

// Recursive integration for the second integral. Left and Right are vectors
// with the lower (resp. upper) bounds in each of the directions.
interval rec_integrate(const ivector& left, const ivector& right,
                          const interval& h2, const interval& Kappa){
  RecCalls++;
  ivector measure; measure = (right-left);
  interval vol(1.0);
  for (int i = 1; i <= VecLen(measure); i++){
    vol*= measure[i];
  }
  int flag = 0;
  interval res = Simpson_integrand_p2(left[1],right[1],left[2],right[2],h2,
                                         Kappa,flag)*vol;
  // We bisect either if we don't meet the tolerances or if we divide by zero
  if (diam(res) > abs_tol || diam(res)/mid(vol) > rel_tol || flag == 1){
    // bisection
    interval new_res(0.0);

    ivector midpoint = (right+left)/2.0;
    for (int i = 0; i < (1 << VecLen(left)); i++){
      ivector curr_left;
      ivector curr_right;
      curr_left = midpoint; curr_right = midpoint;
      for (int j = 0; j < VecLen(left); j++){
        if (i & (1 << j)){
          curr_left[j+1] = left[j+1];
        }
        else {
          curr_right[j+1] = right[j+1];
        }
      }
      new_res+=rec_integrate(curr_left,curr_right,h2,Kappa);
    }
    return new_res;
  }
  else {
    return res;
```

```
  }
}

////////////////////////////////////////////////////////////
// END DEFINITION OF THE INTEGRATION METHODS
////////////////////////////////////////////////////////////

////////////////////////////////////////////////////////////
// BEGIN DEFINITION OF THE MESH GENERATION METHODS
////////////////////////////////////////////////////////////

void build_mesh1(int N, interval Left){
  interval Full; Full = Pi(); Full-= Left;
  for (int i=0;i<=N;i++){
    mesh1[i] = (interval)(i)*Full;
    mesh1[i] /= N;
    mesh1[i] += Left;
  }
}

////////////////////////////////////////////////////////////
// END DEFINITION OF THE MESH GENERATION METHODS
////////////////////////////////////////////////////////////


int main(int argc, char* argv[]){
  cout << SetPrecision(29,26);
  if (argc != 2) {
    cout << "Usage: " << endl;
    cout << argv[0] << " N" << endl;
    cout << "N = Number of intervals in which we discretize the first (1D) integral"
           << endl;
    exit(0);
  }
  int N = atoi(argv[1]);
  // Neighbourhood of the singularity set to 1/128 which is representable exactly.
  interval Left; "[0.0078125,0.0078125]" >> Left;
  // Boundary for the bounded part of the second interval
  interval Right = (interval)Pi_interval;
  build_mesh1(N, Left);
  interval h2 = Pid2_interval;
  interval Kappa; "[1.0,1.0]" >> Kappa;
  interval resP1, resP2; resP1 = 0.0; resP2 = 0.0;
  int flag = 0;

  // Computation of the outer factor \da z_{2}(0)
  itaylor i_zero = itaylor(1,0.0);
  itaylor Z2 = z2(i_zero);
  interval DZ20 = get_j_coef(Z2,1);
```

```
  // Computation of the first integral
  for (int i = 0; i < N; i++){
    interval dx; dx = mesh1[i+1] - mesh1[i];
    interval IP1;
    IP1 = Simpson_integrand_p1(mesh1[i],mesh1[i+1],flag);
    resP1+=IP1*dx;
  }
  resP1 = resP1*DZ20;
  if (flag == 1){
    cout << "Division by zero at the first integral. Set a finer mesh." << endl;
    return 0;
  }
  interval sing1; sing1 = 0.0;
  sing1 = singularity_p1(Left|0.0,flag)*Left;
  sing1 = sing1*DZ20;
  if (flag == 1){
    cout << "Division by zero at the singularity. Set 'Left' closer to 0." << endl;
    return 0;
  }

  // Computation of the second integral
  ivector left(2); left[1] = 0.0; left[2] = -Pi_interval;
  ivector right(2); right[1] = Right; right[2] = Pi_interval;
  resP2 = rec_integrate(left,right,h2,Kappa);
  resP2 = resP2*DZ20;

  cout << "RESULTS" << endl << endl;
  cout << "First = " << resP1 + sing1 << endl;
  cout << "Second = " << resP2 << endl;
  cout << "Total = " << resP1 + sing1 + resP2 << endl;
  cout << endl;
  cout << "DIAMETERS" << endl << endl;
  cout << diam(resP1+sing1) << endl;
  cout << diam(resP2) << endl;
  cout << diam(resP1 + sing1 + resP2) << endl;

  cout << "Number of recursive calls = " << RecCalls << endl;

  return 0;
}
```

B.4 Theorem 6.2.3

Listing B.8: Proof of Theorem 6.2.3

```
// Proof of Theorem 6.2.3(a/b). Case Flat at infinity.
//
```

```
// We use this program to validate two scenarios. In the first one, the
// contribution of the first term is negative, and the contribution of the
// second one is positive, but with an absolute value less than the first
// one (for Kappa = -1). This shows that regardless of the value of Kappa,
// the curve will turn over.
//
// In the second scenario, the contribution of the first integral cannot
// dominate the contribution of the second one. Hence, there exist values
// of Kappa for which the total sum is negative (and therefore the curve
// turns over) and others for which the total sum is positive (and the curve
// does not turn over).
//
// The first integral is a one-dimensional integral and is split into two
// parts, one corresponding to a singularity (of type 0/0, although continuous
// in the limit) and another corresponding to a smooth integrand.
// This integral (except the singular part) is computed using Simpson's
// rule on a uniform mesh of N1 subintervals, where N1 is specified by the user.
//
// The second integral is a two-dimensional integral and is also split into two
// parts, one corresponding to an unbounded region in which we perform
// theoretical estimates to bound the tails (see equation XXX) which
// are partly estimated using the computer (functions and terms "Tails").
// The tails are 1D integrals and are estimated using a 1D Simpson's rule on a
// uniform mesh of N2 subintervals, being N2 also specified by the user.
// The second part corresponds to a bounded region where the integrand is smooth.
// The integration is done in an adaptive way, using a 2D Simpson's rule and
// stopping the integration when the relative and absolute width of the results
// meet some tolerance criteria, otherwise splitting into four regions, cutting
// in each dimension in half. We also keep track of the number of recursive calls
// (i.e the number of boxes in which we integrate).
//
// In order to speed up some calculations, some functions are duplicated
// depending on the input/output data types (either interval, itaylor or dim2taylor).
// For the sake of clarity, we discarded the use of templates by duplicating code.
//
// All calculations are done using rigorous arithmetics using the C-XSC library.

// Usage: ./2D_v4 N1 N2
// N1 = Number of intervals in which we discretize the first (1D) integral
// N2 = Number of intervals in which we discretize the tail estimates for the second
// (2D) integral

// Output (Case a)
// RESULTS (N1 = 8192, N2 = 256, RelTol = 1e-5, AbsTol = 1e-5, Kappa = -1)

// First = [-0.745640133646000013989407777,-0.745639899943032258278208224]
// Tail Estimates = [ 0.00002666357059504430928069, 0.00002667699868569642701806]
// Tail = [-0.00002667699868569642701806, 0.00002667699868569642701806]
// Second = [ 0.02034657516565453738710544, 0.02068402037491086017939602]
```

```
// Total = [-0.72532023547903123894542433,-0.72492920256943560453066766]
//
// DIAMETERS
//
// 0.00000023370296775571119952
// 0.00000001342809065211773736
// 0.00005335399737139285403610
// 0.00033744520925632279229056
// 0.00039103290959563441475666
// Number of recursive calls = 9205


// Output (Case b)
// RESULTS (N1 = 8192, N2 = 256, RelTol = 1e-5, AbsTol = 1e-5, Kappa = 1)

// First = [-0.00059107053222070204349243,-0.00059083812284237949459531]
// Tail Estimates = [ 0.00002519841429958738886619, 0.00002520339771374320749073]
// Tail = [-0.00002520339771374320749073, 0.00002520339771374320749073]
// Second = [ 0.00839958726488038848190242, 0.00871016631351071227151728]
// Total = [ 0.00778331333494594254651666, 0.00814453158838207742775684]
//
// DIAMETERS
//
// 0.00000023240937832254889711
// 0.00000000498341415581862453
// 0.00005040679542748641498145
// 0.00031057904863032378961485
// 0.00036121825343613488124017
// Number of recursive calls = 9421

#include "interval.hpp"
#include "itaylor.hpp"
#include "dim2taylor.hpp"

#include <iostream>

#define MAXN 500000
using namespace cxsc;
using namespace std;
using namespace taylor;

// Maximum order of the Taylor expansion for the Newton-Cotes quadratures.
#define ORDER_TAYLOR 4
// Maximum order of the Taylor expansion of the singularity at 0.
#define ORDER_TAYLOR_SINGULARITY 6

// Mesh for the first integral
interval mesh1[MAXN];
// Mesh for the tail estimates of the second integral
```

```
interval mesh2[MAXN];

// Tolerances for the adaptive integration
const double abs_tol = 1e-5;
const double rel_tol = 1e-5;

// Number of recursive calls in the adaptive integration
int RecCalls = 0;


/////////////////////////////////////////////////////////
// DEFINITION OF THE CURVES
/////////////////////////////////////////////////////////

// Parametrization of the curve: itaylor data type
itaylor z1(const itaylor& x, const interval& K){
  return x - sin(x)*exp(-K*sqr(x));
}

// Comment or uncomment lines 2/4 depending on the version of the Theorem (a/b)
// one wants to validate.
itaylor z2(const itaylor& x){
  // Case turning independent of Kappa
  return sin((interval)3.0*x)/(interval)3.0 - sin(x)*(exp(-sqr(x+(interval)2.0))
                  +exp(-sqr(x-(interval)2.0)));
  // Case turning dependent of Kappa
  // return sin((interval)1.88*x)/(interval)1.88 - (interval)1.0*sin(x)
                  *(exp(-sqr(x+(interval)2.0))+exp(-sqr(x-(interval)2.0)));
}

// Parametrization of the curve: interval data type coming from ivector
// This is useful to get fast 0-th order representations of z1 while
// working on the second (2D) integral
interval z1(const ivector& v, const interval& K){
  return v[2] - sin(v[2])*exp(-K*sqr(v[2]));
}

// Comment or uncomment lines 2/4 depending on the version of the Theorem (a/b)
// one wants to validate.
interval z2(const ivector& v){
  // Case turning independent of Kappa
  return sin((interval)3.0*v[2])/(interval)3.0
                  - sin(v[2])*(exp(-sqr(v[2]+(interval)2))
                  +exp(-sqr(v[2]-(interval)2)));
  // Case turning dependent of Kappa
  // return sin((interval)1.88*v[2])/(interval)1.88
                  - (interval)1.0*sin(v[2])*(exp(-sqr(v[2]+(interval)2))
                  +exp(-sqr(v[2]-(interval)2)));
}
```

```
// Parametrization of the curve: interval data type coming from interval
interval z1(const interval& v, const interval& K){
  return v - sin(v)*exp(-K*sqr(v));
}

// Comment or uncomment lines 2/4 depending on the version of the Theorem (a/b)
// one wants to validate.
interval z2(const interval& v){
  // Case turning independent of Kappa
  return sin((interval)3.0*v)/(interval)3.0 - sin(v)*(exp(-sqr(v+(interval)2))
                 +exp(-sqr(v-(interval)2)));
  // Case turning dependent of Kappa
  // return sin((interval)1.88*v)/(interval)1.88
                 - (interval)1.0*sin(v)*(exp(-sqr(v+(interval)2))
                 +exp(-sqr(v-(interval)2)));
}

// Parametrization of the curve: dim2taylor data type
dim2taylor z1(const dim2taylor_vector& v, const interval& K){
  return v[2] - sin(v[2])*exp(-K*sqr(v[2]));
}

// Comment or uncomment lines 2/4 depending on the version of the Theorem (a/b)
// one wants to validate.
dim2taylor z2(const dim2taylor_vector& v){
  // Case turning independent of Kappa
  return sin((interval)3.0*v[2])/(interval)3.0
                 - sin(v[2])*(exp(-sqr(v[2]+(interval)2))
                 +exp(-sqr(v[2]-(interval)2)));
  // Case turning dependent of Kappa
  // return sin((interval)1.88*v[2])/(interval)1.88 - (interval)1.0*sin(v[2])
                 *(exp(-sqr(v[2]+(interval)2))+exp(-sqr(v[2]-(interval)2)));
}

///////////////////////////////////////////////////////////
// END DEFINITION OF THE CURVES
///////////////////////////////////////////////////////////


///////////////////////////////////////////////////////////
// BEGIN DEFINITION OF THE INTEGRANDS
///////////////////////////////////////////////////////////

// Singular part of the first integral
interval singularity_p1(const interval& alpha, const interval& K, int& flag){
  itaylor x(ORDER_TAYLOR_SINGULARITY+1,alpha);
  itaylor Z1 = z1(x,K); itaylor DZ1 = diff(Z1,1);
  Z1 = trunc(Z1,ORDER_TAYLOR_SINGULARITY);
```

```
  x = trunc(x,ORDER_TAYLOR_SINGULARITY);
  itaylor Z2 = z2(x);
  // taylor on the numerator
  itaylor num = 4.0*Z1*Z2*DZ1;
  // taylor on the denominator
  itaylor den = sqr(sqr(Z1)+sqr(Z2));
  if (0 <= get_j_coef(den,4)){
    // Division by zero
    flag = 1;
    return interval(0.0);
  }
  // Den is O(alpha^4), Num is O(alpha^6) when expanded around alpha = 0;
  return get_j_coef(num,6)/get_j_coef(den,4)*sqr(alpha);
}


// Integrand of the first integral: itaylor data type
itaylor integrand_p1(const interval& x, const interval& K, int order, int& flag){
  itaylor x_it = itaylor(order+1,x);
  itaylor Z1 = z1(x_it,K);
  itaylor DZ1 = diff(Z1, 1);
  Z1 = trunc(Z1,order);
  x_it = trunc(x_it,order);
  itaylor Z2 = z2(x_it);
  // Now, Z1 and DZ1 are 'order'-order itaylor objects
  itaylor den = (sqr(sqr(Z1)+sqr(Z2)));
  if (0 <= get_j_coef(den,0)){
    // Division by zero
    flag = 1;
    return itaylor(0);
  }
  return (interval)4.0*DZ1*Z1*Z2/den;
}


// Integrand of the first integral: interval data type
interval integrand_p1(const interval& x, const interval& K, int& flag){
  itaylor x_it = itaylor(1,x);
  itaylor Z1 = z1(x_it,K);
  itaylor DZ1 = diff(Z1, 1);
  Z1 = trunc(Z1,0);
  interval Z2 = z2(x);
  // Now, Z1 and DZ1 are 0-order itaylor objects
  itaylor den = (sqr(sqr(Z1)+sqr(Z2)));
  if (0.0 <= get_j_coef(den,0)){
    // Division by zero
    flag = 1;
    return interval(0.0);
  }
```

```
  itaylor integrand = (interval)4.0*DZ1*Z1*Z2/den;
  return get_j_derive(integrand,0);
}

// Integrand of the second integral: dim2taylor data type
dim2taylor integrand_p2(ivector& iv, const interval& K, const interval& h2,
                            const interval& Kappa, int order, int& flag){
  dim2taylor_vector dv; dv = init_var(order+1,iv);
  dim2taylor_vector dv_low; dv_low = init_var(order,iv);
  dim2taylor Z2 = z2(dv);
  dim2taylor DZ2 = diff(Z2,0,1);
  Z2 = trunc(Z2,order);
  dim2taylor Z1 = z1(dv_low,K);
  dim2taylor den1 = sqr(sqr(dv_low[1])+sqr(h2));
  dim2taylor den2 = sqr(h2+Z2)+sqr(dv_low[1]-Z1);
  dim2taylor den3 = sqr(h2+Z2)+sqr(-dv_low[1]-Z1);
  if (0.0 <= den1[0][0] || 0.0 <= den2[0][0] || 0.0 <= den3[0][0]){
    // Division by zero
    flag = 1;
    return dim2taylor(0);
  }
  return -Kappa/Pi_interval*(sqr(dv_low[1]))/den1*((DZ2)*(h2+Z2))
         *((interval)1.0/den2+(interval)1.0/den3);
}

// Integrand of the second integral: interval data type
interval integrand_p2(const ivector& iv, const interval& K, const interval& h2,
                            const interval& Kappa, int& flag){
  itaylor x_it = itaylor(1,iv[2]);
  itaylor Z2 = z2(x_it);
  //itaylor DZ2 = diff(Z2, 1);
  interval DZ2 = get_j_derive(Z2,1);
  Z2 = trunc(Z2,0);
  interval Z1 = z1(iv[2],K);
  interval den1 = sqr(sqr(iv[1])+sqr(h2));
  itaylor den2 = sqr(h2+Z2)+sqr(iv[1]-Z1);
  itaylor den3 = sqr(h2+Z2)+sqr(-iv[1]-Z1);
  if (0 <= den1 || 0.0 <= get_j_coef(den2,0) || 0.0 <= get_j_coef(den3,0)){
    // Division by zero
    flag = 1;
    return interval(0.0);
  }
  // Now, Z2 and DZ2 are 0-order itaylor objects
  itaylor integrand = -Kappa/Pi_interval*(sqr(iv[1]))/den1*((DZ2)*(h2+Z2))
                      *((interval)1.0/den2+(interval)1.0/den3);
  return get_j_derive(integrand,0);
}

// Integrand of the tail of the second integral: itaylor data type
```

```
itaylor integrand_tail(const interval& beta, const interval& x, const interval& K,
                       const interval& h2, const interval& Kappa, int order, int& flag){
  itaylor x_it = itaylor(order+1,x);
  itaylor Z2 = z2(x_it);
  itaylor DZ2 = diff(Z2, 1);
  Z2 = trunc(Z2,order);
  x_it = trunc(x_it,order);
  itaylor Z1 = z1(x_it,K);
  // Now, Z2 and DZ2 are 'order'-order itaylor objects
  itaylor den1 = sqr(h2+Z2)+sqr(beta-Z1);
  itaylor den2 = sqr(h2+Z2)+sqr(-beta-Z1);
  if (0.0 <= get_j_coef(den1,0) || 0.0 <= get_j_coef(den2,0)){
    // Division by zero
    flag = 1;
    return itaylor(0);
  }
  return -Kappa/Pi_interval*((DZ2)*(h2+Z2))*((interval)1.0/den1+(interval)1.0/den2);
}


// Integrand of the tail of the second integral: interval data type
interval integrand_tail(const interval& beta, const interval& x, const interval& K,
                        const interval& h2, const interval& Kappa, int& flag){
  itaylor x_it = itaylor(1,x);
  itaylor Z2 = z2(x_it);
  itaylor DZ2 = diff(Z2, 1);
  Z2 = trunc(Z2,0);
  interval Z1 = z1(x,K);
  // Now, Z2 and DZ2 are 0-order itaylor objects
  itaylor den1 = sqr(h2+Z2)+sqr(beta-Z1);
  itaylor den2 = sqr(h2+Z2)+sqr(-beta-Z1);
  if (0.0 <= get_j_coef(den1,0) || 0.0 <= get_j_coef(den2,0)){
    flag = 1;
    return interval(0);
  }
  itaylor integrand = -Kappa/Pi_interval*((DZ2)*(h2+Z2))
                      *((interval)1.0/den1+(interval)1.0/den2);
  return get_j_derive(integrand,0);
}

/////////////////////////////////////////////////////////////
// END DEFINITION OF THE INTEGRANDS
/////////////////////////////////////////////////////////////


/////////////////////////////////////////////////////////////
// BEGIN DEFINITION OF THE INTEGRATION METHODS
/////////////////////////////////////////////////////////////
```

```
// Simpson's method for the first integral
interval Simpson_integrand_p1(const interval& left_abscissa,
                              const interval& right_abscissa,
                              const interval& K, int& flag){
  interval mid_interval = (left_abscissa+right_abscissa)/(interval)2.0;
  interval alpha = left_abscissa|right_abscissa;
  interval dx = right_abscissa - left_abscissa;
  interval left = integrand_p1(left_abscissa,K,flag);
  interval mid = integrand_p1(mid_interval,K,flag);
  interval right = integrand_p1(right_abscissa,K,flag);
  itaylor error = integrand_p1(alpha,K,ORDER_TAYLOR,flag);
  return (left+(interval)4.0*mid+right-sqr(sqr(dx))/(interval)20.0
              *get_j_coef(error,4))/(interval)6.0;
}


// Simpson's two-dimensional method for the second integral.
// We assume the integrand is written with the variables ordered as F(beta,gamma)
interval Simpson_integrand_p2(const interval& beta_left, const interval& beta_right,
                              const interval& gamma_down, const interval& gamma_up,
                              const interval& K, const interval& h2,
                              const interval& Kappa,int& flag){
  // Simpson's rule with 9 points
  // Left Up
  ivector left_up_i(2);
  left_up_i[1] = beta_left; left_up_i[2] = gamma_up;
  interval left_up = integrand_p2(left_up_i,K,h2,Kappa,flag);


  // Center Up
  ivector center_up_i(2);
  center_up_i[1] = (beta_left+beta_right)/(interval)2.0;
  center_up_i[2] = gamma_up;
  interval center_up = integrand_p2(center_up_i,K,h2,Kappa,flag);


  // Right Up
  ivector right_up_i(2);
  right_up_i[1] = beta_right; right_up_i[2] = gamma_up;
  interval right_up = integrand_p2(right_up_i,K,h2,Kappa,flag);


  // Left Center
  ivector left_center_i(2);
  left_center_i[1] = beta_left;
  left_center_i[2] = (gamma_up+gamma_down)/(interval)2.0;
  interval left_center = integrand_p2(left_center_i,K,h2,Kappa,flag);
```

```
// Center Center
ivector center_center_i(2);
center_center_i[1] = (beta_left+beta_right)/(interval)2.0;
center_center_i[2] = (gamma_up+gamma_down)/(interval)2.0;
interval center_center = integrand_p2(center_center_i,K,h2,Kappa,flag);


// Right Center
ivector right_center_i(2);
right_center_i[1] = beta_right;
right_center_i[2] = (gamma_up+gamma_down)/(interval)2.0;
interval right_center = integrand_p2(right_center_i,K,h2,Kappa,flag);


// Left Down
ivector left_down_i(2);
left_down_i[1] = beta_left; left_down_i[2] = gamma_down;
interval left_down = integrand_p2(left_down_i,K,h2,Kappa,flag);


// Center Down
ivector center_down_i(2);
center_down_i[1] = (beta_left+beta_right)/(interval)2.0;
center_down_i[2] = gamma_down;
interval center_down = integrand_p2(center_down_i,K,h2,Kappa,flag);


// Right Down
ivector right_down_i(2);
right_down_i[1] = beta_right; right_down_i[2] = gamma_down;
interval right_down = integrand_p2(right_down_i,K,h2,Kappa,flag);


// Error
ivector error_i(2);
error_i[1] = beta_left|beta_right; error_i[2] = gamma_up|gamma_down;
dim2taylor error = integrand_p2(error_i,K,h2,Kappa,ORDER_TAYLOR,flag);

if (flag == 1){
  return interval(0.0);
}

interval dx = beta_right - beta_left;
interval dy = gamma_up - gamma_down;

interval error_4x = error[4][0];
interval error_4y = error[0][4];

return ((interval)16*center_center
```

```
          +(interval)4*(center_up+left_center+right_center+center_down)
          +left_up+right_up+left_down+right_down)/(interval)36.0
          -(error_4x*sqr(sqr(dx))+error_4y*sqr(sqr(dy)))/(interval)120.0;
}


// Computes the tail (L^1 integral in gamma) for a given beta.
// Since the integral is C^0 but not C^1 at some points, we use quadrature of
// order 0 when the function is not differentiable (i.e. when 0 is contained
// in the integrand). Otherwise we use Simpson.
interval Simpson_integrand_tail(const interval& beta, const interval& left_abscissa,
                                const interval& right_abscissa, const interval& K,
                                const interval& h2, const interval& Kappa, int& flag){
  interval alpha = left_abscissa|right_abscissa;
  itaylor error = integrand_tail(beta,alpha,K,h2,Kappa,ORDER_TAYLOR,flag);
  if (0 <= get_j_coef(error,0)){
    return abs(get_j_coef(error,0));
  }
  else {
    interval mid_interval = (left_abscissa+right_abscissa)/(interval)2.0;
    interval dx = right_abscissa - left_abscissa;
    interval left = integrand_tail(beta,left_abscissa,K,h2,Kappa,flag);
    interval mid = integrand_tail(beta,mid_interval,K,h2,Kappa,flag);
    interval right = integrand_tail(beta,right_abscissa,K,h2,Kappa,flag);
    return abs(left+(interval)4.0*mid+right-sqr(sqr(dx))/(interval)20.0
                    *get_j_coef(error,4))/(interval)6.0;
   }
}

// Recursive integration for the second integral. Left and Right are vectors
// with the lower (resp. upper) bounds in each of the directions.
interval rec_integrate(const ivector& left, const ivector& right,
                       const interval& K,const interval& h2,
                       const interval& Kappa){
  RecCalls++;
  ivector measure; measure = (right-left);
  interval vol(1.0);
  for (int i = 1; i <= VecLen(measure); i++){
    vol*= measure[i];
  }
  int flag = 0;
  interval res = Simpson_integrand_p2(left[1],right[1],left[2],
                                      right[2],K,h2,Kappa,flag)*vol;
  if (diam(res) > abs_tol || diam(res)/mid(vol) > rel_tol || flag == 1){
    // bisection
    interval new_res(0.0);

    ivector midpoint = (right+left)/2.0;
    for (int i = 0; i < (1 << VecLen(left)); i++){
```

```
      ivector curr_left;
      ivector curr_right;
      curr_left = midpoint; curr_right = midpoint;
      for (int j = 0; j < VecLen(left); j++){
        if (i & (1 << j)){
          curr_left[j+1] = left[j+1];
        }
        else {
          curr_right[j+1] = right[j+1];
        }
      }
      new_res+=rec_integrate(curr_left,curr_right,K,h2,Kappa);
    }
    return new_res;
  }
  else {
    return res;
  }
}

//////////////////////////////////////////////////////////////////
// END DEFINITION OF THE INTEGRATION METHODS
//////////////////////////////////////////////////////////////////

//////////////////////////////////////////////////////////////////
// BEGIN DEFINITION OF THE MESH GENERATION METHODS
//////////////////////////////////////////////////////////////////

// Builds the mesh for the first integral
void build_mesh1(int N, interval Left){
  interval Full; Full = Pi(); Full-= Left;
  for (int i=0;i<=N;i++){
    mesh1[i] = (interval)(i)*Full;
    mesh1[i] /= N;
    mesh1[i] += Left;
  }
}

// Builds the mesh for the estimates of the tail of the second integral
void build_mesh2(int N){
  interval Full; Full = Pi_interval;
  for (int i=0;i<=N;i++){
    mesh2[i] = (interval)(i)*Full;
    mesh2[i] /= N;
  }
}

//////////////////////////////////////////////////////////////////
// END DEFINITION OF THE MESH GENERATION METHODS
```

```
//////////////////////////////////////////////////////////////


int main(int argc, char* argv[]){
  cout << SetPrecision(29,26);
  if (argc != 3) {
    cout << "Usage: " << endl;
    cout << argv[0] << " N1 N2" << endl;
    cout << "N1 = Number of intervals in which we discretize the negative
                (1D) integral" << endl;
    cout << "N2 = Number of intervals in which we discretize the tail estimates
                for the positive (2D) integral" << endl;
    exit(0);
  }
  int N1 = atoi(argv[1]);
  int N2 = atoi(argv[2]);
  // Neighbourhood of the singularity set to 1/128 which is representable exactly.
  interval Left; "[0.0078125,0.0078125]" >> Left;
  // Boundary for the tail estimates
  interval Right = (interval)7.0*Pi2_interval;

  build_mesh1(N1, Left);
  build_mesh2(N2);
  interval K; "[0.01,0.01]" >> K;
  interval h2 = Pid2_interval;
  interval Kappa(-1.0);
  interval resP1, resP2; resP1 = 0.0; resP2 = 0.0;
  int flag = 0;

  // Computation of the outer factor \da z_{2}(0)
  itaylor i_zero = itaylor(1,0.0);
  itaylor Z2 = z2(i_zero);
  interval DZ20 = get_j_coef(Z2,1);


  // Computation of the First integral
  for (int i = 0; i < N1; i++){
      interval dx; dx = mesh1[i+1] - mesh1[i];
      interval IP1;
      IP1 = Simpson_integrand_p1(mesh1[i],mesh1[i+1],K,flag);
      resP1+=IP1*dx;
  }
  resP1 = resP1*DZ20;
  if (flag == 1){
    cout << "Division by zero at the first integral. Set a finer mesh." << endl;
    return 0;
  }
```

```
interval sing1; sing1 = 0.0;
sing1 = singularity_p1(Left|0.0,K,flag)*Left;
sing1 = sing1*DZ20;
if (flag == 1){
  cout << "Division by zero at the singularity. Set 'Left' closer to 0." << endl;
  return 0;
}

// Computation of the Second integral
ivector left(2); left[1] = 0.0; left[2] = -Pi_interval;
ivector right(2); right[1] = Right; right[2] = Pi_interval;
resP2 = rec_integrate(left,right,K,h2,Kappa);
resP2 = resP2*DZ20;

// Computation of the tail estimates
interval tailEstimates; tailEstimates = 0.0;
for (int i = 0; i < N2; i++){
  interval dy; dy = mesh2[i+1] - mesh2[i];
  interval IPtail = Simpson_integrand_tail(Right, mesh2[i], mesh2[i+1],K,
                                            h2,Kappa,flag);
  tailEstimates+=IPtail*dy;
}
tailEstimates = tailEstimates*DZ20;
if (flag == 1){
  cout << "Division by zero at the tail. Set 'Right' closer to infinity." << endl;
  return 0;
}

// We multiply the estimates on the tail by
// \int_{Right}^{\infty} \frac{\beta^{2}}{(\beta^{2}+h_{2}^{2})^{2}}d\beta
tailEstimates = tailEstimates *
               (Pi_interval/((interval)4.0*h2) - atan(Right/h2)/((interval)2*h2)
               + Right/((interval)2*(sqr(h2)+sqr(Right))));

interval tail = tailEstimates | -tailEstimates;

// Output:
cout << "RESULTS" << endl << endl;
cout << "First = " << resP1 + sing1 << endl;
cout << "Tail Estimates = " << tailEstimates << endl;
cout << "Tail = " << tail << endl;
cout << "Second = " << resP2 << endl;
cout << "Total = " << resP1 + sing1 + resP2 + tail << endl;

cout << endl;

cout << "DIAMETERS" << endl << endl;
cout << diam(resP1+sing1) << endl;
cout << diam(tailEstimates) << endl;
```

```
  cout << diam(tail) << endl;
  cout << diam(resP2) << endl;
  cout << diam(resP1 + sing1 + resP2 + tail) << endl;

  cout << "Number of recursive calls = " << RecCalls << endl;

  return 0;
}
```

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